JOURNAL OF ALGEBRA 48, 417–438 (1977)

Standard Subgroups of Type $L_n(2^a)$

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Communicated by Marshall Hall, Jr.

Received September 22, 1976

INTRODUCTION

In this paper we prove the following classification theorem.

THEOREM. *Let G be a finite group and let A be a standard subgroup of G. Assume that $|Z(A)|$ is odd, $C_G(A)$ has cyclic Sylow 2-subgroups, and $\tilde{A} = A/Z(A) \simeq L_n(q)$, q even, where $n \geqslant 3$ and $\tilde{A} \not\simeq L_3(2), L_4(2)$. In addition we assume*

(*) *In core-free sections of G the 2-layers of centralizers of 2-subgroups are semisimple.*

Then one of the following holds:

(i) $O(G)A \trianglelefteq G$,

(ii) $E(G/Z(G)) \simeq L_n(q^2)$.

(iii) $E(G/Z(G)) \simeq L_n(q) \times L_n(q)$.

We remark that $(*)$ is needed only when G is of type (iii) in order to show that a certain subgroup G_0 with $G_0/Z(G_0) \simeq L_n(q) \times L_n(q)$ is, in fact, normal in G. Here it is used only when the 2-components involved are of type $L_n(q)$. Finally, condition $(*)$ is a consequence of the B-conjecture of Thompson.

Combining the above theorem with the work in [1, 6, 7, 18] we have the following.

COROLLARY. *Let G be a finite group with standard subgroup A satisfying $|Z(A)|$ odd and $A/Z(A) \simeq L_n(q)$. In addition assume condition $(*)$. Then one of the following holds:*

(i) $O(G)A \trianglelefteq G$.

(ii) $E(G/Z(G)) \simeq L_n(q^2)$.

(iii) $E(G/Z(G)) \simeq L_n(q) \times L_n(q)$.

(iv) $E(G/Z(G)) \simeq U_3(q)$, $n = 2$.

* Supported in part by the NSF under Grant GP 37982X1.

ISSN 0021-8693

(v) $E(G/Z(G)) \cong L_3(q)$, $n = 2$.

(vi) $E(G/Z(G)) \cong J_2$ *and* $A \cong L_3(2)$.

(vii) $E(G/Z(G)) \cong HS$, A_{10}, *or* A_{12}, *and* $A \cong L_4(2)$.

(viii) $E(G/Z(G)) \cong M_{12}$, A_9, J_1 *or* J_2, *and* $A \cong L_2(4)$.

(ix) $E(G/Z(G)) \cong G_2(3)$ *and* $A \cong L_2(8)$.

(x) $E(G/Z(G)) \cong Sz$ *and* $A \cong L_3(4)$.

(xi) $E(G/Z(G)) \cong L_2(25)$, $U_3(5)$, *or* $L_3(5)$, *and* $A \cong L_2(4)$.

In all cases other than (i), (vii), (viii), and (x), $C_G(A)$ has Sylow 2-subgroups of order 2. This also occurs in (vii) in $E(G/Z(G)) \cong HS$ or A_{10} and in (viii) if $E(G/Z(G)) \cong M_{12}$ and $G/O(G) \not\cong \mathrm{Aut}(M_{12})$. In the remaining cases $C_G(A)$ has Sylow 2-subgroups as Klein groups.

The case of $L_n(q)$ has been separated from the work in [16] for the following reason. The induction method of [16] breaks down in low dimensions, so that in any case $L_3(q)$, $L_4(q)$, and $L_5(q)$ would have to be treated separately. However, the alternate method used for the cases $n \leqslant 5$ works equally well for any $n \geqslant 3$. The method is to use the parabolic subgroups of A corresponding to the stabilizer of a nested 1-space and hyperplane of the usual n-dimensional $\mathbb{F}_q$-space for $SL(n, q)$ in order to construct the corresponding parabolic subgroups of the group $L_n(q^2)$ or $L_n(q) \times L_n(q)$. At this stage it is clear which of (ii) or (iii) holds. For (ii) we use McBride's result [12] to obtain the conclusion. For (iii) we first construct a group G_0 with $G_0/Z(G_0) \cong L_n(q) \times L_n(q)$ and then use the results of [16, I, Sect. 2]. In independent work Gomi has used similar methods to deal with the case $A/Z(A) \cong L_3(q)$, $q > 4$.

Throughout the paper the groups A, G are as in the hypothesis of the theorem. Given a group H we set $\tilde{H} = H/Z(H)$. All groups considered are finite.

1. Preliminary Lemmas

(1.1) *Let* $G_0 \cong L_n(q^2)$ *with* $n \geqslant 3$. *Let* σ *be an involutory field automorphism of* G_0 *and let* τ *be a graph automorphism of* G_0.

(i) $O^{2'}(C_{G_0}(\sigma)) \cong L_n(q)$, $O^{2'}(C_{G_0}(\tau)) \cong PSp(n, q^2)$ *if* n *is even*, $O^{2'}(C_{G_0}(\tau)) \cong PSp(n-1, q^2)$ *if* n *is odd*, $O^{2'}(C_{G_0}(\sigma\tau)) \cong PSU(n, q)$.

(ii) *If* n *is odd, then each involution in* $\mathrm{Aut}(G_0) - G_0$ *is conjugate to one of* σ, τ, $\sigma\tau$.

(iii) *If* n *is even, then each involution in* $\mathrm{Aut}(G_0) - G_0$ *is conjugate to one of* σ, τ, $\sigma\tau$, *or* τx, *where* $x \in G_0$ *is a transvection in* G_0. *Moreover* $C_{G_0}(\tau x)$ *is isomorphic to the centralizer of a transvection in* $PSp(n, q^2)$.

Proof. See [2, Sect. 19].

(1.2) *Let T be a p-group and x an element of order p acting on T. If $1 = T_1 < T_1 < \cdots < T_n = T$ is a normal series of T with each T_{i+1}/T_i elementary and free as an $\mathbb{F}_p(\langle x\rangle)$-module, then $|C_T(x)| = |T|^{1/p}$ and each subgroup of order p in $T\langle x\rangle - T$ is conjugate to $\langle x\rangle$.*

Proof. See [20].

(1.3) *Let G be a group and let A be a standard subgroup of G with $C_G(A)$ of 2-rank 1. Let $S \in \mathrm{Syl}_2(N(A))$. Assume that $S \notin \mathrm{Syl}_2(G)$ and $Z(S) \leqslant AC_G(A)$. Then $[A, O(G)] = 1$ so any 2-component of G containing A is necessarily a component of G.*

Proof. Given G, A, S let $t \in S \cap C_G(A)$ with t an involution. Choose $g \in N_G(S) - S$ with $g^2 \in S$. Then $t^g \in Z(S)$ but $t^g \neq t$. Let $tt^g = a$ and $Y = \langle t, t^g\rangle$. Write $X = O(G)$ and $X = C_X(t)\, C_X(ta)\, C_X(a)$.

Now $C_X(t) \leqslant N(A)$, so $[A, C_X(t)] \leqslant A \cap X$ and $[A, C_X(t)] = 1$. In particular $C_X(t) \leqslant C_X(Y)$. As $Y = Y^g$ we also have $C_X(t^g) \leqslant C_X(Y)$. We conclude that $a \in C_G(X) = C_G(O(G))$, so $C_G(O(G)) \geqslant \langle a^G\rangle \geqslant A$.

2. Local Subgroups

In this section we determine the structure of certain local subgroups of G. We assume that A is a standard subgroup of G with $|Z(A)|$ odd and $\tilde{A} = A/Z(A) \simeq L_n(q)$. We assume that G is a minimal counterexample to the main theorem. Then $AO(G) \ntrianglelefteq G$. Recall that $\tilde{A} \not\simeq L_3(2)$ or $L_4(2)$.

Let $K = C_G(A)$, $R \in \mathrm{Syl}_2(K)$, and $V \in \mathrm{Syl}_2(A)$. We are assuming R is cyclic, so let $\langle t\rangle = \Omega_1(R)$ and write $U = Z(V)$. Then U is elementary Abelian of order q. Finally let $S \in \mathrm{Syl}_2(N(A))$ with $VR \leqslant S$.

(2.1) (i) $S \notin \mathrm{Syl}_2(G)$.

(ii) $R = \langle t\rangle$ *and* $Ut \subseteq t^G$.

(iii) *If* $t \neq t^g \in Z(S)$, *then* $V \leqslant A^g$.

(iv) $t^G \cap U = \varnothing$.

Proof. We first claim that $Ut \subseteq t^G$. As A is transitive on its transvections we need show only that there is some $t^g \in A^{\#}t$ with t^g projecting to a transvection in A. If $\tilde{A} \simeq L_3(q)$ this follows from [16, I, (1.2)] and the fact that A has just one class of involutions. For $n \geqslant 4$ let X and D_{00} be as in [16, I, (1.6)]. Then $\tilde{D}_{00} \simeq L_{n-2}(q)$ is simple and [16, I, (1.4)] gives $t^G \cap A\langle t\rangle \cap C(X) \neq \varnothing$. Say $n = 4$. Then let $t \neq t^g \in A\langle t\rangle \cap C(X)$; so t^g projects to a transvection in A. If $t^g \in A$, then $t^g \in C_A(t^g)^{(\infty)}$, whereas $t \notin C_G(t)^{(\infty)}$. So we are done if $n = 4$. Now suppose $n \geqslant 5$, set $Y = O(C_G(X))$, and let bars denote images in $C_G(X)/Y$. By [16, I, (1.10)] $\overline{D_{00}}$ is standard but not normal in $\overline{C_G(X)}$. By the minimality of G and the corollary we known the structure of $E(\overline{C_G(X)})$. In all but one case it is easy to check that the claim holds. The exceptional case is where $\overline{D_{00}} \cong L_4(2) \cong A_8$ and $\overline{E(C_G(X))} \cong A_{10}$. Here t acts on $\overline{E(C_G(X))}$ as a transposition. Identify t with $(1, 2)$ and $\overline{D}_{00}$ with the subgroup of A_{10} fixing points 1 and 2. From [16, I, (1.4)] we have $t^h \in D_{00}^{\#}\langle t\rangle$ for some $h \in G$. Say t^h projects to $a \in D_{00}$. Then a is either a transvection or has precisely two nontrivial Jordan blocks (viewing $a \in SL(6, 2)$). In either case, $a \in C_A(a)'$. Since $\tilde{A} \cong L_6(2)$ we have $t \notin C_G(t)'$, so $t^G \cap D_{00} = \varnothing$ and $t^h = ta$. We are done if a is a transvection so suppose this is not the case. Viewing $\bar{a} \in \overline{D_{00}} \cong A_8$ we may identify $\bar{a}$ with $(3, 4)(5, 6)$ and $\overline{t^h}$ with $(1, 2)(3, 4)(5, 6)$. Viewing $C_G(X) \cap C(t^h)$ as a subgroup of $C(X)$ and then of $C(t^h)$ we obtain a contradiction. The claim follows.

Choose $g \in G$ with $t^g \in Z(S)$. Since $S \in \mathrm{Syl}_2(C(t))$ we may assume that $g \in N(S)$. Notice that $V \in \mathrm{Syl}_2(O^2(C(Z(S))))$ so $g \in N(V)$. In particular $V \leqslant A^g$, proving (iii). We know that $N_A(U)$ is transitive on $U^{\#}$ so if (iv) were false then it would be possible to choose $t^g \in U \cap Z(S)$. But then $t^g \in V \leqslant A^g$ implies that $t \in A$, whereas $|Z(A)|$ is odd. So we also have (iv).

Let $N = N(S)$. We have seen that $N \leqslant N(V)$, so N normalizes each of $Z(V) = U$ and $C_S(V) = U \times R$. If $|R| > 2$, then t is the unique involution in $\Phi(C_S(V))$, so $N \leqslant C(t)$, which is not the case. This proves (ii).

Consider the action of N on $U \times R$. We know that $t^G \cap (U \times R) = t^N \cap (U \times R) = Ut$. As $N_A(U)$ is transitive on $U^{\#}$, N acts on Ut as a 2-transitive group of degree q and $S \in \mathrm{Syl}_2(C_N(t))$. In particular (i) holds.

(2.2) *Notation.* Let $\hat{}$ denote images in $A/Z(A)$. Let $V_1 = O_2(P_1)$ and $V_2 = O_2(P_2)$, where $\hat{P}_1$, $\hat{P}_2$ are, respectively, the stabilizer of a 1-space, hyperplane, of the natural module for $L_n(q)$. We take $V \leqslant P_1 \cap P_2$, so $V_i \leqslant V$. Then $\widehat{N_A(V_i)} \trianglerighteq \hat{V}_i(\hat{L}_i \circ \hat{H}_i)$, where $\hat{L}_i \cong SL(n-1, q)$ and $\hat{H}_i$ is cyclic of order dividing $q - 1$ and divisible by $(1/n)(q-1)$. Also, $S \cap N_A(V_i) = VR\langle w\rangle$, where w induces a field automorphism on A.

(2.3) *Let X act faithfully on the elementary Abelian group $W \times \langle t\rangle$, normalizing W and 2-transitive on Wt. Suppose that $C_X(t)$ contains a normal subgroup $K \cong SL(m, q)$, $m \geqslant 2$ and $q = 2^a$, such that K acts on W as on the usual module for $SL(m, q)$. Then X contains a normal elementary subgroup of order q^m.*

Proof. Using the result of Ostrom and Wagner [15] we may assume $m \geqslant 3$. The assumption on W and K immediately implies that $|W| = q^m$ and $|X : C_X(t)| = q^m$. Let $S \in \mathrm{Syl}_2(C_X(t))$ and $S < \bar{S} \in \mathrm{Syl}_2(X)$. Then $X = \bar{S}C_X(t)$. Choose $w \in W^{\#}$ with $\bar{S} \leqslant C(w)$ and let W_0 be the corresponding 1-space of W containing w, when W is viewed as the usual module for K. Then $S \cap K = C_S(W_0)$ and $W_0 = C_W(S \cap K)$.

Consider the group $Y = N_X(W_0\langle t\rangle)$. We know that $Y \cap K$ induces a solvable group on W_0 of order $(q-1)b$ with $b \mid a$ and is transitive on $W_0{}^{\#}$. So either $Y \leqslant C(t)$ or Y is 2-transitive on W_0t of degree q. In the latter case look at the group $C_Y(C(W_0\langle t\rangle)/O_2C(W_0\langle t\rangle)) = Y_0$. It is easy to see that $Y_0 \cap C(t)$ is regular on $W_0t - \{t\}$, so Y_0 must induce a Frobenius group of order $q(q-1)$ on $W_0\langle t\rangle$. As $C_K(W_0\langle t\rangle)$ covers $C(W_0\langle t\rangle)/O_2(C(W_0\langle t\rangle))$ and is irreducible on W/W_0, we must have $Y \leqslant (Y \cap C(t))\, C(W/W_0)$.

We claim that $N_X(W_0)$ contains a Sylow 2-subgroup of X. We may assume that $\bar{S} \cap N_X(W_0) \in \mathrm{Syl}_2(N_X(W_0))$. First suppose $\bar{S} \cap N_X(W_0) = \bar{S} \cap Y$. By the above it follows that $\bar{S} \cap Y = (\bar{S} \cap C(W_0))\langle j\rangle$, where $j \in C_X(t)$ and induces a field automorphism of K. Say $g \in N_{\bar{S}}(\bar{S} \cap Y) - (\bar{S} \cap Y)$. Let $H = \bar{S} \cap K$. Then $W_0 = C_W(H) = C_W(\bar{S} \cap C(W_0))$, so $H^g \nleqslant \bar{S} \cap C(W_0)$. But then $\langle j\rangle \neq 1$, $q \geqslant 4$, and $H^g = \Omega_1(H^2)$ has a subgroup of index 2 centralizing W_0. This forces $W_0 = C_W(H^g)$ and $g \in N(W_0)$, a contradiction. So we have the claim in this case. Now suppose $\bar{S} \cap N_X(W_0) > \bar{S} \cap Y$. Then we use the above paragraph in order to apply induction to the group $N_X(W_0)$ acting on $W \times \langle t\rangle/W_0 = W/W_0 \times \langle t\rangle W_0/W_0$. As $N_K(W_0)$ is transitive on $(W/W_0)^{\#}$ and since we are assuming that $N_X(W_0) \nleqslant N(W_0\langle t\rangle)$, we have a 2-transitive group on $(W/W_0)\, tW_0$. Inductively this group is 2-transitive of degree q^{m-1} and contains a regular normal subgroup. But then $\bar{S} \cap N_X(W_0) = (\bar{S} \cap C(W_0))\langle j\rangle$ for $j \in C_X(t)$ a field automorphism of K, and we argue as above to get $\bar{S} = N_{\bar{S}}(\bar{S} \cap N(W_0))$. This proves the claim.

To complete the proof of the lemma we note that $X = \bar{S}C_X(t) = N(W_0)C_X(t)$ implies that X is 2-transitive on $W_0{}^X = W_0^{C_X(t)}$. Write $N_X(W_0) = \bar{S}(N_X(W_0) \cap C_X(t))$. Then intersecting the conjugates of $\bar{S}$ in $N_X(W_0)$ we have $O_2(N_X(W_0)) \neq 1$. So by the results of O'Nan [13] and Shult [17] we get the lemma.

(2.4) *$N_G(V_i\langle t\rangle)$ normalizes V_i and acts on tV_i as a 2-transitive group of degree q^{n-1} containing a regular normal subgroup.*

Proof. This follows immediately from (2.3) once we show that $X_i = N_G(V_i\langle t\rangle)$ normalizes V_i and is 2-transitive on tV_i. $V_i^{\#}$ is fused in A, so by (2.1)(iv) $t^G \cap V_i = \varnothing$. Suppose we have $X_i \nleqslant C(t)$. Then as $N_A(V_i)$ is transitive on $V_i^{\#}$, $t^G \cap V_i\langle t\rangle = tV_i$, $V_i = V_i\langle t\rangle - t^G \trianglelefteq X_i$, and X_i is 2-transitive on tV_i. So we need only show $X_i \nleqslant C(t)$.

Let $t^g \in Ut - \{t\}$, so by (2.1)(iv), $V \leqslant A^g$. If $n = 3$ then V_1, V_2 are uniquely determined as the elementary subgroups of V of maximal order. So V_i is A^g-conjugate to $V_1{}^g$ or $V_2{}^g$, and so $N_{A^g}(V_i) \nleqslant C(t)$. If $n \geqslant 4$, then $V_1V_2 = O_2(C_A(t^g)) \leqslant (C_A(t^g)^{(\infty)}) < C(t^g)^{(\infty)} \leqslant A^g$. Again V_1, V_2 are uniquely determined and we can argue as before.

(2.5) *Let bars denote images in $N_G(V_i\langle t\rangle)/O(N_G(V_i\langle t\rangle))$. Then $\overline{N_G(V_i\langle t\rangle)} = \bar{W}_i\bar{L}_i\langle \bar{t}, \bar{w}\rangle$ where $W_i > V_i$ is an L_i-invariant elementary Abelian 2-group, W_i/V_i is elementary of order q^{n-1}, and $W_i/V_i \cong V_i$ as L_i-modules.*

Proof. By (2.4) $\overline{N_G(V_i\langle t\rangle)}$ contains a normal 2-subgroup $\bar{X}$. We choose X a 2-group with $V_i\langle t\rangle < X$. $N_A(V_i)$ contains a cyclic subgroup Y_0 with Y_0 irreducible on $V_i^{\#}$. The map $\bar{x} \to [\bar{x}, t]$ gives an $N_A(V_i)$-isomorphism between

$\bar{X}/\overline{V_i\langle t\rangle}$ and $\bar{V}_i$. Therefore $\widehat{N_A(V_i)}$ is transitive on $(\bar{X}/\overline{V_i\langle t\rangle})^{\#}$ and it follows that $\bar{X}/\bar{V}_i$ is elementary Abelian. Let $\bar{W}_i = [\bar{X}, Y_0]$. Then Y_0 acts irreducibly on $\bar{W}_i/\bar{V}_i$ and on $\bar{V}_i$.

We claim that $\bar{W}_i$ is Abelian. Easily $\bar{V}_i \leqslant Z(\bar{W}_i)$. In many cases $\widehat{N_A(V_i)}$ contains a subgroup $Y \geqslant Y_0$ cyclic of order $q^{n-1} - 1$. In such cases Y stabilizes $\bar{W}_i = [\bar{X}, Y_0]$. Moreover the map $\bar{w}_i \to [\bar{w}_i, t]$ is a Y-homomorphism from $\bar{W}_i$ to $\bar{V}_i$ with kernel $\bar{V}_i$. So the claim follows from [6, (2.2)]. Suppose now that no such subgroup, Y, exists. In particular $q > 2$ and n divides $q - 1$; so $n = 3$ or $n \geqslant 5$.

Let $\bar{Z}/\overline{V_i\langle t\rangle}$ be a $GF(q)$-hyperplane in $\bar{X}/\overline{V_i\langle t\rangle}$. Then $\bar{L}_i$ contains a subgroup D cyclic of order $q^{n-2} - 1$ with D regular on $(\bar{Z}/\overline{V_i\langle t\rangle})^{\#}$ and transitive on $(\bar{X}/\bar{Z})^{\#}$. We write $\bar{V}_i = \bar{Z}_0 \times \bar{Z}_1$ where $\bar{Z}_0 = [\bar{Z}, t]$ and $\bar{Z}_1$ is D-invariant. Then $\bar{Z}_0$ is a hyperplane in $\bar{V}_i$, $\bar{Z}_1$ is a 1-space, and D is transitive on each of $\bar{Z}_0$ and $\bar{Z}_1$. Next write $\bar{Z}/\bar{V}_i = \bar{C}/\bar{V}_i \times \overline{\langle t\rangle V_i}/\bar{V}_i$ with $\bar{C}$ D-invariant, and consider $\bar{C}/\bar{Z}_0$ and $\bar{C}/\bar{Z}_1$. $\bar{C}/\bar{Z}_1$ is Abelian by [6, (2.2)]. Suppose $n \geqslant 5$. To see that $\bar{C}/\bar{Z}_0$ is Abelian notice that $| \bar{C}/\bar{V}_i = q^{n-2} \geqslant q^3$. As $n \geqslant 5$ $\bar{C}/\bar{Z}_0$ cannot be a Suzuki 2-group [8], so as $\bar{D}$ is transitive on $(\bar{C}/\bar{V}_i)^{\#}$, $\bar{C}/\bar{Z}_0$ is actually elementary Abelian. In all cases each of $\bar{C}/\bar{Z}_1$ and $\bar{C}/\bar{Z}_0$ are Abelian, which yields $\bar{C}$ Abelian. By order considerations we see that $\bar{C}$ is uniquely determined by $\bar{Z}$.

Choose $g \in L_i$ with $\langle \bar{C}, \bar{C}^g\rangle$ covering $\bar{X}/\overline{V_i\langle t\rangle}$. Let $\hat{W}_i = \langle \bar{C}, \bar{C}^g\rangle$. Then $\hat{W}_i$ centralizes $\overline{C \cap C^g}$ and $\hat{W}_i$ has index of at most 2 in $\bar{X}$. Moreover by order considerations $\overline{C \cap C^g} > \bar{V}_i$. Since $Z(\bar{X}) = \bar{V}_i$ we have $\hat{W}_i$ of index 2 in $\bar{X}$. If $h \in N_A(V_i)$ then we again use order considerations to argue that $\overline{C^g \cap C \cap C^h} > \bar{V}_i$. So the above arguments imply that $\langle \bar{C}, \bar{C}^g, \bar{C}^h\rangle \neq \bar{X}$. We conclude that $\hat{W}_i$ is normalized by $N_A(V_i)$. Therefore $\hat{W}_i = [\bar{X}, N_A(V_i)]$, and as $\bar{W}_i = [\bar{X}, Y_0]$ we must have $\bar{W}_i = \hat{W}_i$. As Y_0 acts irreducibly on $\bar{W}_i/\bar{V}_i$ and $\bar{V}_i < \overline{C \cap C^g} \leqslant Z(\bar{W}_i)$ we must have $\bar{W}_i$ Abelian as claimed.

Now we consider the case $n = 3$. First suppose that $q \neq 4$. Then $\overline{N_A(V_i)}$ contains a cyclic subgroup $1 \neq Y_1$ of order $q - 1$ or $\frac{1}{3}(q - 1)$ with $[L_i, Y_1] = 1$. Then $[\bar{X}, Y_1]$ is L_i-invariant and we conclude that $[\bar{X}, Y_1] = \bar{W}_i^{L_i}$. We have $\bar{W}_i/\bar{V}_i$ isomorphic to $\bar{V}_i$ as L_i-modules. Let $\bar{Z}/\bar{V}_i$ be a 1-space of $\bar{W}_i/\bar{V}$.

Let Z_0, Z_1 be as before, so that $\bar{C}/\bar{Z}_1$ is abelian. If $q = 4$ we also have $\bar{C}/\bar{Z}_0$ abelian by [6, (2.2)]. So in this case $\bar{C}$ is abelian. Suppose $q > 4$ and let $P = L_i \cap V$. We may assume P centralizes $C/\bar{V}_i$, so P centralizes $\bar{W}_i/C$, $\bar{Z}_0$, and $\bar{V}_i/\bar{Z}_0$. Note that $[\bar{W}_i, P]$ covers $\bar{C}/\bar{Z}_0$. Now $[\bar{W}_i, [\bar{W}_i, P], P] \leqslant [\bar{V}_i, P] \leqslant \bar{Z}_0$, and $[[\bar{W}_i, P], P, \bar{W}_i] \leqslant [\bar{V}_i, \bar{W}_i] = 1 \leqslant \bar{Z}_0$. Consequently $[[\bar{W}_i, P], [\bar{W}_i, P]] \leqslant Z_0$ and again $\bar{C}/\bar{Z}_0$ is abelian. Also P centralizing $\bar{W}_i/\bar{C}$ and $\bar{C}/\bar{V}_i$ implies P centralizes $[\bar{C}, \bar{W}_i]$. Thus $[\bar{C}, \bar{W}_i] \leqslant \bar{Z}_0$. Now let $\bar{I}/\bar{V}_i$ be a 1-space of $\bar{W}_i/\bar{V}_i$ complementing $\bar{C}/\bar{V}_i$ and let $\bar{I}_0 = [\bar{I}, t]$. Then symmetry gives $[\bar{I}, \bar{C}] \leqslant \bar{I}_0 \cap \bar{Z}_0 = 1$. Since $W_i = \langle \bar{C}, \bar{I}\rangle$ we have the claim.

Finally we consider the case $n = 3$ and $q = 4$. Let $\bar{Y} = [\bar{X}, D] = \bar{Y}_1/\bar{V}_i \times \bar{Y}_2/\bar{V}_i$ with each factor D-invariant of order 4. As before $\bar{Y}_1$ and $\bar{Y}_2$ are abelian. But since Y_0 is fixed-point-free on $\bar{W}_i$, an easy calculation in the Lie ring (or use of commutators) shows $\bar{Y}' < \bar{V}_i$. If $\bar{Y}' = 1$ we argue that $\bar{Y}$ is characteristic in $\bar{X}$, forcing $\bar{Y} = \bar{W}_i$. Suppose $\bar{Y}' \neq 1$. Then regarding $\bar{V}_i$ as an $\mathbb{F}_2$-space of dimension 4, $\bar{Y}'$ is a 2-space in $\bar{V}_i$. Moreover we may choose $g \in L_i$ with $Y' \cap (Y')^g = 1$. But consider $Y \cap Y^g$. This group is of index at most 2 in Y. But $(Y \cap Y^g)' \leqslant Y' \cap (Y')^g = 1$, so $(Y \cap Y^g)' = 1$. One checks that Y can have no such subgroup. This contradiction completes the proof of the claim.

Suppose $g \in N(V_i\langle t\rangle) \leqslant N(V_i)$. Then $\bar{W}_i \cap \bar{W}_i^g$ has index of at most 2 in $\bar{W}_i$ and is centralized by $\langle \bar{W}_i, \bar{W}_i^g\rangle$. As $C_X(t) = V_i\langle t\rangle$ we must have $\bar{W}_i = \bar{W}_i^g$. So $\bar{W}_i \trianglelefteq \overline{N_G(V_i)}$. Next write $N(V_i\langle t\rangle) = O(N(V_i\langle t\rangle))(N(V_i\langle t\rangle) \cap N(X))$. Since $N(V_i\langle t\rangle) \cap N(X) \leqslant XC(t)$, we have $L_i \leqslant N(V_i\langle t\rangle) \cap N(X)$, and we conclude that $L_i \leqslant N(W_i)$.

It remains to show that W_i is elementary Abelian. Suppose not. Then W_i is homocyclic of exponent 4 and we may take $i = 1$. Notice that if $S_1 \in \mathrm{Syl}_2$ $(N_G(V_1)) \cap N_G(W_1\langle t\rangle))$ then $[S_1, t] = V_1$. Let $g \in N(S_1)$. Then $W_1^g < S_1$ and $[W_1^g, t] \leqslant W_1^g \cap V_1$ is elementary Abelian. So $t \in C_{S_1}(W_1^g/V_1^g) = W_1^g\langle t^g\rangle$. Also $t \notin S'_1$, so $t \notin V_1^g$. Therefore $t \in W_1^g t^g$ and so $[S_1, t] \geqslant V_1^g$. By order consideration $V_1 = V_1^g$. So $g \in N(V_1)$, $g \in N(C_S(V_1)) = N(W_1\langle t\rangle)$, and $S_1 \in \mathrm{Syl}_2(G)$.

Now S_1 normalizes V_2^k, W_2^k, and $(W_2\langle t\rangle)^k$ for some $k \in G$. Suppose that W_2 is elementary Abelian. Then $[W_2^k, W_1] \leqslant W_2^k \cap W_1 \leqslant V_1$, so W_2^k centralizes W_1/V_1 forcing $W_2^k \leqslant W_1\langle t\rangle$, which is impossible. Thus W_2 is also homocyclic of exponent 4. As above $[S_1, t] = V_1$ implies that $t \in W_2^k t^k$ and $V_2^k = V_1$. Then $W_2^k = W_1$, S_1 and $S_1^k \leqslant N(V_1)$, and we may take $k \in N(S_1)$. This implies that $W_2 = W_1^{k^{-1}} \trianglelefteq S_1$ and $V_2 \trianglelefteq S_1$. But then $[V_2, W_1] \leqslant V_2 \cap W_1 \leqslant V_1$, whereas V_2 does not centralize W_1/V_1. This contradiction establishes (2.5).

(2.6) *Let* $N_i = N_G(W_i)$ *and* $C_i = V \cap L_i$. *Then* $S_i = W_iC_i\langle w, t\rangle \notin \mathrm{Syl}_2(N_i)$.

Proof. Suppose that $S_i \in \mathrm{Syl}_2(N_i)$. We may take $i = 1$. First assume that $n = 3$. We claim that W_1 is weakly closed in S_1. For suppose $W_1 \neq W_1^g \nleqslant S_1$. The elementary Abelian subgroups of maximal order in S_1/W_1 have order $2q$, while $|W_1^g| = q^4$. Since $q > 2$, $|W_1^g \cap W_1| > q^2$. Viewing W_1/V_1 and V_1 as $\mathbb{F}_q$-modules, this implies that either $W_1^g \cap V_1$ generates V_1 or $(W_1^g \cap W_1) V_1/V_1$ generates W_1/V_1. But W_1^g centralizes $W_1^g \cap W_1$, so $W_1^g \cap W_1C_1\langle t\rangle$ centralizes W_1/V_1 or V_1, forcing $W_1^g \cap W_1C_1\langle t\rangle \leqslant W_1\langle t\rangle$. Then $W_1^g \leqslant C(W_1^g \cap W_1C_1\langle t\rangle)$ implies $W_1^g \leqslant W_1\langle t\rangle$. As all involutions in $W_1\langle t\rangle$ are in W_1 or V_1t, $W_1^g = W_1$, a contradiction. This proves the claim. In particular, since we are assuming $S_1 \in \mathrm{Syl}_2(N(W_1))$, we must have $S_1 \in \mathrm{Syl}_2(G)$. We can now obtain a contradiction by applying [4, Theorem 4]. Let $q = 2^a > 2$. The number r in Goldschmidt's result is at most $a + 1$ as $L_1 \cong SL(2, q)$. Let $x \in S_1 - W_1$ be an involution. Then $[W_1, x] \geqslant q^2 = 2^{2a} > 2^r$. So by [4, Theorem 4], W_1 is strongly closed in S_1. The main theorem of [4] gives a contradiction.

We now assume that $n \geqslant 4$. First we construct a certain subgroup of G, and for this we introduce notation from the (B, N) structure of A. Let Σ be a root system of type A_{n-1} with base $\pi = \{\alpha_1, ..., \alpha_{n-1}\}$. We may assume that V is written $V = \prod_{r \in \Sigma^+} U_r$ and generated by the root subgroups $U_{\alpha_1}, ..., U_{\alpha_{n-1}}$. In this notation we may assume $V_1 = U_{\alpha_1} U_{\alpha_1+\alpha_2} U_{\alpha_1+\cdots+\alpha_{n-1}}$ and $L_1 = \langle U_{\pm\alpha_2}, ..., U_{\pm\alpha_{n-1}} \rangle$. If $n \geqslant 5$ set $X = \langle U_{\pm\alpha_2} \rangle \cong SL(2, q)$ and if $n = 4$ let $X = A \cap C(\langle U_{\pm\alpha_3}, U_{\pm(\alpha_1+\alpha_2)} \rangle)$. In the latter case X is cyclic of order $q - 1 > 1$. Viewing L_1 as matrices (via the modules V_1 and W_1/V_1) operating on the right we have

$$C_1 = \left\{ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \right\},$$

$$U_{\alpha_2} = \left\{ \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & a & 1 \end{pmatrix} : a \in \mathbb{F}_q \right\}, ..., U_{\alpha_{n-1}} = \left\{ \begin{pmatrix} 1 & & & \\ a & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} : a \in \mathbb{F}_q \right\}.$$

In addition, if $n \geqslant 5$ we set

$$J = \left\{ \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ * & \cdots & * & 1 & \\ * & \cdots & * & 0 & 1 \end{pmatrix} \right\} \quad \text{and} \quad F = \left\{ \begin{pmatrix} I & & \\ & 1 & \\ & & 1 \end{pmatrix} : I \in SL(n-3, q) \right\}.$$

Finally if $n = 4$, then X induces

$$\left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & \alpha \end{pmatrix} : \alpha \in \mathbb{F}_q^{\#} \right\}$$

on V_1 and we set

$$J = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ * & * & 1 \end{pmatrix} \right\}, \qquad F = \left\{ \begin{pmatrix} I & \\ & 1 \end{pmatrix} : I \in SL(2, q) \right\}.$$

Notice that $F = \langle U_{\pm\alpha_4}, ..., U_{\pm\alpha_{n-1}} \rangle$. Now the construction of X forces $X \leqslant C(U_{\alpha_1+\cdots+\alpha_{n-1}})$ so X centralizes an involution $t \neq t^g \in tU_{\alpha_1+\cdots+\alpha_{n-1}}$. We note that $E(C(X) \cap C(t)) \cong SL(n-2, q)$ if $n \geqslant 5$ and $E(C(X) \cap C(t)) \cong SL(3, q)$ if $n = 4$. Moreover $E(C(X) \cap C(t))$ is standard in $C(X)$ and $\langle t \rangle \in \mathrm{Syl}_2(C(X) \cap C(E(C(X))))$. Inductively either $t \in Z^*(C(X))$, $A \cong L_5(2)$ and

$E(C(X)) \cong J_2$, $A \cong L_6(2)$ and $E(C(X)) \cong A_{10}$, or $E(C(X))/Z(E(C(X))) \cong L(m, q^2)$ or $L(m, q) \times L(m, q)$, where $E(C_A(X))/Z(E(C_A(X))) \cong L(m, q)$. However, we know that $X \leqslant C_A(t^g) = C_A(t^g)^{(\infty)} \leqslant N(A^g)^{(\infty)} = A^g$, and consideration of $C_{A^g}(X)$ shows that $t \notin Z^*(C_{A^g}(X))$. Consequently $t \notin Z^*(C(X))$.

If $n = 4$ let $X_1 = X$ and if $n \geqslant 5$ let $X_1 \leqslant X$ be a cyclic group of order $q + 1$. Write $V_0 = C_{V_1}(X_1)$ and note that as X acts in the same way on W_1/V_1 and on V_1, $|C_{W_1}(X_1)| = |V_0|^2$. In case $n \geqslant 5$ the above arguments show that $E(C_G(X_1)) = E(C_G(X))$, so $W_0 = C_{W_1}(X_1) \leqslant E(C_G(X))$. An easy argument shows $W_0 \nleqslant S_{10}$. Suppose $E(C(X)) \cong J_2$. The Sylow 2-subgroups of J_2 are isomorphic to the Sylow 2-subgroup of $L(3, 4)\langle\sigma\rangle$ for σ a graph-field automorphism (see [21, (10.1)]. Let $I \in \mathrm{Syl}_2(E(C(X)))$ with $t \in N(I)$ and $W_0 \leqslant I$. Now W_0 is elementary of order 2^4 and $W_0{}^t = W_0$, so $W_0 \leqslant I_0 \in \mathrm{Syl}_2(L(3, 4))$, $I_0 \trianglelefteq I\langle t\rangle$. Looking in $N_G(V_1) \cap C(X)$ we see that V_0 is contained in $E(C(X\langle t\rangle))$, so from [21, (10.1), (4), (5)] we conclude that $V_0 = Z(I_0)$. Then $I_0\langle t\rangle = I\langle t\rangle \cap C(V_0)$ and choosing $h \in I - I_0$ we have $[W_0, t] = V_0 = [W_0{}^h, t]$. But then $[t, I_0] = V_0$ and $|E(C(X)) \cap C(t) \cap C(V_0)| \geqslant 2^4$, against the structure of $E(C(X)) \cap C(t) \cong PGL(2, 7)$.

If $n = 4$, $V_0 = U_{\alpha_1+\alpha_2}U_{\alpha_1+\alpha_2+\alpha_3}$ and if $n \geqslant 5$, $V_0 = U_{\alpha_1+\alpha_2+\alpha_3} \cdots U_{\alpha_1+\cdots+\alpha_{n-1}}$. V_0 is the unique hyperplane, respectively $((n-1)-2)$-space, of V_1 normalized by S_1. Then $V_0 \leqslant E(C(X) \cap C(t))$ and by order $O_2(C(X) \cap C(V_0)) = W_0$. Moreover writing $|V_0| = q^b$, $N(W_0) \cap E(C(X))$ induces $SL(b, q^2)$ or $SL(b, q) \times SL(b, q)$ on W_0.

We claim that $W_0 = C_{W_1}(X)$ and $W_0 \trianglelefteq S_1$. By earlier remarks $W_0 \leqslant C_{W_1}(X) \leqslant C_{W_1}(X_1) = W_0$, so we have the first part of the claim.

For the second part of the claim we must show that $W_0 \trianglelefteq S_1$. First assume that $n \geqslant 5$. Note that $X \times F$ acts on W_1 and on $W_0 = C_{W_1}(X)$. Also $[X_1, W_1]$ is F-invariant and F centralizes $[X_1, W_1]/[X_1, V_1]$ and also $[X_1, V_1]$. Therefore $O^2(F)$ centralizes $[X_1, W_1]$. Except in the case $A \cong L_5(2)$ this implies that $F \leqslant C([X_1, W_1])$. Assume also that $A \ncong L_5(2)$. Then $U_{\alpha_4} \leqslant F$ centralizes $[X_1, W_1]$ and from the structure of $E(C(X)) \cap N(W_0)$, $|[U_{\alpha_4}, W_0]| = q^2$. So $|[U_{\alpha_4}, W_1]| = q^2$, as $W_1 = W_0 \times [W_1, X_1]$. Now L_1 is transitive on its root subgroups. J centralizes V_0, V_1/V_0, and W_0V_1/V_1. If $\alpha \in \Sigma^+$ with $U_\alpha \leqslant J$, then by the above $[W_1, U_\alpha] \cap V_1 \leqslant V_0$. In particular $[W_0, U_\alpha] \leqslant V_0$ and $U_\alpha \leqslant N(W_0)$. As J is a product of such subgroups U_α, we have $J \leqslant N(W_0)$. Hence $C_1 \leqslant N(W_0)$. As $S_1 = W_1C_1\langle w, t\rangle$ and w normalizes X, we have $W_0 \trianglelefteq S_1$, proving the claim in the cases $n \geqslant 5$ and $A \ncong L_5(2)$.

Suppose that $A \cong L_5(2)$. As above we only need show that $|[U_{\alpha_4}, W_1]| = q^2 = 4$. Since U_{α_2} and U_{α_4} are L_1-conjugate it suffices to show that $|[U_{\alpha_2}, W_1]| = 4$. Now $D = \langle U_{\pm\alpha_2}\rangle \cong S_3$ and $W_1 = W_0 \times [W_1, O_3(D)]$. On the second factor $\langle U_{\pm\alpha_2}\rangle = U_{\alpha_2}O_3(D)$ acts with $O_3(D)$ fixed-point-free. It follows that U_{α_2} induces two copies of the regular representation on $[W_1, O_3(D)]$. As $U_{\alpha_2} \leqslant C(W_0)$, $|[U_{\alpha_2}, W_1]| = 4$ as needed.

Finally we take the case $n = 4$. Write $W_1 = W_0 \times [W_1, X]$ and note that here $[W_1, X]$ has order q^2. Also $\langle U_{\pm\alpha_3}\rangle$ centralizes $[W_1, X]/[V_1, X]$ and $[V_1, X]$, so $\langle U_{\pm\alpha_3}\rangle = O^2(\langle U_{\pm\alpha_3}\rangle) \leqslant C([W_1, X])$. Therefore $|[W_1, U_{\alpha_3}]| = q^2$ and we argue as before. At this point the claim is proved.

Consider $N_G(W_0) \geqslant S_1$. Then $C(V_0) \geqslant JX$ and $[J, X] = J$. As $X \leqslant C(W_0)$, it follows that $J \leqslant C(W_0)$. So S_1 induces $(S_1 \cap F)\langle w_1, t\rangle$ on W_0.

Assume that $n \geqslant 5$. Also assume that $S_1 \cap C(W_0) \notin \mathrm{Syl}_2(C(W_0))$. Then there is an element $g \in N(S_1) - S_1$ such that $g \in C(W_0)$ and $g^2 \in S_1$. So $V_0 = V_0{}^g$ and $[t^g, S_1] = V_1{}^g \geqslant V_0$. If $V_1{}^g \leqslant W_1$, then $W_1 \leqslant C_{S_1}(V_1{}^g) = W_1{}^g$, so $g \in N(W_1)$, against our hypothesis that $S_1 \in \mathrm{Syl}_2(N(W_1))$. Therefore $V_1{}^g \nleqslant W_1$. Now $t^g \in S_1 \cap C(V_0) = \langle t\rangle W_1 J U_{\alpha_2}$. Also $t^g \notin W_1 t$ or else $g \in W_1 C(t)$. Say t^g induces an involution on W_1/V_1 having l nontrivial Jordan blocks. Then $|[t^g, W_1/V_1]| = q^l$, and since $V_0 \leqslant [t^g, S_1] \leqslant V_1{}^g \nleqslant W_1$, $l = 1$. Also $[t^g, S_1] \lhd S_1$, so $([t^g, S_1] \cap W_1) V_1/V_1 \trianglelefteq S_1/V_1$. It follows that $[t^g, W_1] V_1/V_1$ is the unique 1-space of W_1/V_1 fixed by S_1. Now $W_1{}^g \leqslant S_1 \cap C(V_0) = W_1 J U_{\alpha_2}$, $[W_1{}^g, t^g] = V_1{}^g$, and $W_1{}^g \trianglelefteq S_1$. It follows that $W_1{}^g \nleqslant W_1 J$ and so $[W_1{}^g, W_1/V_1]$ is a hyperplane in W_1/V_1. Similarly $[W_1{}^g, V_1]$ is a hyperplane in V_1. As $[W_1{}^g, W_1] \leqslant W_1 \cap W_1{}^g$, $|W_1 \cap W_1{}^g| \geqslant q^{2(n-2)}$. Hence $|W_1{}^g W_1/W_1| \leqslant q^2$. But $|[W_1{}^g, S_1] W_1/W_1| \geqslant |J|^{1/2} = q^{n-3} \geqslant q^2$, and since $W_1{}^g W_1/W_1 > [W_1{}^g, S_1] W_1/W_1$, this is a contradiction. Therefore $S_1 \cap C(W_0) \in \mathrm{Syl}_2(C(W_0))$.

However, $S_1 \notin \mathrm{Syl}_2(N(W_0))$. In fact $N(W_0) \cap C(X)$ contains a $\langle t, w\rangle$-invariant subgroup H with H inducing $SL(n-3, q^2)$ or $SL(n-3, q) \times SL(n-3, q)$ on W_0. Then $S_1 \in C(W_0) H\langle t, w\rangle$ and there is an element $g \in N(S_1) \cap N(W_0)$ with $g \notin S_1$, $g^2 \in S_1$, and $t^g \in tU^{\#}_{\alpha_3}(S_1 \cap C(W_0))$. As before, we must have $V_1{}^g \nleqslant W_1$ and $([S_1, t^g] \cap W_1) V_1/V_1 \trianglelefteq S_1/V_1$. It follows that $[S_1, t^g] V_1/V_1 \cap W_1/V_1$ contains an $(n-3)$-subspace of W_1/V_1. Similarly $[S_1, t^g] \cap V_1$ contains an $(n-3)$-subspace of V_1. So $|V_1{}^g \cap W_1| \geqslant q^{2(n-3)} \geqslant q^{n-1} = |V_1| = |V_1{}^g|$. As $V_1{}^g \nleqslant W_1$, this is a contradiction.

Finally consider the case $n = 4$. Choose $g \in N(S_1) \cap N(W_0)$ with $g^2 \in S_1$, but $g \notin S_1$. First assume that $S_1 \cap C(W_0) \notin \mathrm{Syl}_2(C(W_0))$ so that g may be chosen in $C(W_0)$. Then $V_0{}^g = V_0$ and t^g does not centralize W_1/V_1. So $|[t^g, W_1/V_1]| = q$ and $|V_1{}^g \cap W_1| \geqslant |V_0|\, q = q^3 = |V_1{}^g|$. But as before $V_1{}^g \nleqslant W_1$, so this is a contradiction. Therefore $S_0 = S_1 \cap C(W_0) \in \mathrm{Syl}_2(C(W_0))$. We consider the action of $N(S_0) \cap N(W_0)$ on S_0/W_0. We know that $|C_{S_0/W_0}(t)| = q^3$ and $|S_0/W_0| = q^4$. Also $N(W_0) \cap N(S_0)$ contains a subgroup $(B_1 \times B_2)\langle t\rangle$ with $B_1 \cong B_2 \cong SL(2, q)$ and $B_1{}^t = B_2$. So any b_1, b'_1 in B_1, $\langle t, t^{b_1}, t^{b'_1}\rangle$ centralizes a subgroup of order at least q in S_0/W_0. But with proper choice of b_1, b'_1 we have $\langle t, t^{b_1}, t^{b'_1}\rangle = (B_1 \times B_2)\langle t\rangle$. As we easily check that $C_{S_0/W_0}(\langle U_{\pm\alpha_3}\rangle) = 1$, this is impossible. The proof of (2.6) is now complete.

(2.7) *Let* $N_i = N_G(W_i)$.

(i) $O(N_i^{(\infty)}) = Z(N_i^{(\infty)})$.

(ii) *Letting bars denote images in* N_i/W_i, *we have* $E(\overline{N}_i)/Z(E(\overline{N}_i)) \cong L_{n-1}(q^2)$ *or* $L_{n-1}(q) \times L_{n-1}(q)$.

Proof. Consider the group N'_i. We may take $i = 1$. Let $X = O(N'_1)$, let $L_1 = \langle U_{\pm\alpha_1}, \ldots, U_{\pm\alpha_{n-2}} \rangle$ and let z be an involution in $U_{\alpha_1+\cdots+\alpha_{n-2}}$. Then by (2.1) $t \sim tz$, so write $tz = t^g$. Also write $X = C_X(t)\, C_X(tz)\, C_X(z)$. As $C_X(t) \leqslant C(W_1) \leqslant C(V_1)$, $C_X(t) \leqslant O(N(A))$ and hence $C_X(t) \leqslant C(A) \leqslant C(z)$. Next consider $C_X(tz) = C_X(t^g) \leqslant N(A^g)$. By (2.1) the Sylow 2-subgroups of $C_A(t^g)$ are contained in A^g. In particular $C_{V_1}(t^g) = C_{V_1}(z)$ is a hyperplane in V_1 and centralized by $C_X(t^g)$. Since $C_{L_1}(t^g)$ normalizes $C_X(t^g)$ we must have $C_X(t^g) \leqslant O(N(A^g))$ and again $C_X(t^g) \leqslant C(z)$. Therefore X is centralized by z, so $C_{N_1}(X)$, being normal in N_1, must cover $E(\langle z^{N_1} \rangle\, W_1X/W_1X)$.

We know that W_1 controls fusion of $t^G \cap W_1t$. In fact the only involutions in $W_1\langle t \rangle - W_1$ are those in $V_1t = t^{W_1}$. Therefore $C_{\bar{N}_1}(\bar{t}) = \bar{L}_1\langle \bar{w}, \bar{t} \rangle$ and it is easy to see that $\bar{L}_i$ is standard in $\bar{N}_i$. Also $\bar{L}_i \ntrianglelefteq \bar{N}_i$ by (2.6).

By the main theorem in [6] if $n = 3$, Solomon's theorem [17] if $n = 5$ and $q = 2$, or induction for all other values of n and q we conclude that either (2.7) holds, or $n = 3$ and $E(\bar{N}_1) \cong U_3(q), L_3(q), G_2(3), M_{12}$, or $n = 5, q = 2$, and $E(\bar{N}_1) \cong A_{10}$, or HS.

If $E(\bar{N}_1) \cong M_{12}$, then $q = 4$ and $|W_1| = q^4 = 2^8$. But $11 \mid |M_{12}|$ implies that $E(\bar{N}_1)$ must act trivially on W_1, impossible. If $E(\bar{N}_1) \cong U_3(q)$, then by [3, (4*D*)] we get the same contradiction. Suppose $E(\bar{N}_1) \cong L_3(q)$. Here $\bar{t}$ acts as a graph automorphism, so $E(\bar{N}_1)\langle \bar{t} \rangle$ cannot act faithfully on a 2-group of order less than q^6. Again we have a contradiction. If $E(\bar{N}_1) \cong G_2(3)$, then $A \cong L_3(8)$ and $N_A(V_1) \geqslant L_2(8) \times Z_7$. Viewing $N_A(V_1) \leqslant N_A(W_1)$ the Z_7 factor must centralize $E(\bar{N}_1)$ and this leads to $C_2(3) \leqslant L_4(8)$. Impossible.

Suppose $n = 5, q = 2$. If $E(\bar{N}_1) \cong A_{10}$, let $Y \leqslant A$ be a subgroup of order 5. Then $C = C(Y) \cap E(\bar{N}_1)$ involves S_5. But this forces C to be irreducible on W_1 and by Schur's lemma C is isomorphic to a subgroup of $GL(2, 2^4)$. This is impossible. Finally if $E(\bar{N}_1) \cong HS$, then $11 \mid |E(\bar{N}_1)|$. As $|W_1| = 2^8$, this forces $E(\bar{N}_1)$ to act trivially on $|W_1|$, a final contradiction.

3. $\widetilde{E(\bar{N}_i)} \cong L_{n-1}(q^2)$ FOR $i = 1$ OR 2.

In this section we make the assumption that $\widetilde{E(\bar{N}_i)} \cong L_{n-1}(q^2)$ for $i = 1$ or 2, recalling from (2.7) that $\bar{N}_i = N_i/W_i$. We may assume that $i = 1$.

(3.1) *$E(\bar{N}_1) \cong SL(n-1, q^2)$ and W_1 can be viewed as an $\mathbb{F}_{q^2}(E(\bar{N}_1))$-module on which $E(\bar{N}_1)$ induces the usual representation of $SL(n-1, q^2)$. Also $N_1^{(\infty)}$ splits over W_1. In particular the Sylow 2-subgroups of $N_1^{(\infty)}$ are isomorphic to those of $L(n, q^2)$.*

Proof. We begin by proving the first statement, and for this start with the case $n \geqslant 4$. We know that $E(\bar{N}_1)$ is an image of $SL(n-1, q^2)$(23, §8) so consider W_1 as an $\mathbb{F}_2$-module for $SL(n-1, q^2)$. Let $q^2 = 2^a$. By Steinberg [19], $\mathbb{F}_{q^2}$ is a splitting field for $SL(n-1, q^2)$ so we have $W_1 \otimes \mathbb{F}_{q^2} = M_1 \oplus \cdots \oplus M_r$ a sum of inequivalent algebraically conjugate, absolutely irreducible representations of $SL(n-1, q^2)$. If $\theta: x \to x^2$ is the Frobenius map, then M_1 and $M_1^{\theta^r}$ are equivalent and we write $a = rb$. By the tensor product theorem [19]. $M_1 \simeq T_1 \otimes T_2^{\theta} \otimes \cdots \otimes T_a^{\theta^{a-1}}$, where the T_i are absolutely irreducible and we may assume T_1 is nontrivial. As $M_1 \simeq M_1^{\theta^r}$, $\dim(M_1) \geqslant (\dim(T_1))^b$. Also since the Sylow 2-subgroups of $SL(n-1, q^2)$ has class $n-2$, $\dim(T_1) \geqslant n-1$. We now have $a(n-1) = \dim_2(W_1) = \dim_{\mathbb{F}_{q^2}}(W_1 \otimes \mathbb{F}_{q^2}) \geqslant r(\dim(T_1))^b \geqslant r(n-1)^b$. So $b(n-1) \geqslant (n-1)^b$, and as $n-1 \geqslant 3$ we conclude that $b = 1$ and $W_1 \otimes \mathbb{F}_{q^2} = M_1 \oplus M_1^{\theta} \oplus \cdots \oplus M_1^{\theta^{a-1}}$, where M_1 is the usual module for $SL(n-1, q^2)$. If V is the usual module for $SL(n-1, q^2)$, viewed as an $\mathbb{F}_2$-module, then $V \otimes \mathbb{F}_{q^2}$ is also isomorphic to $M_1 \oplus \cdots \oplus M_1^{\theta^{a-1}}$. It follows that $V \simeq W_1$ as $\mathbb{F}_2(SL(n-1, q^2))$ modules and the result holds.

Now suppose $n = 3$. The above arguments carry over and the relevant inequality is $b2 \geqslant 2^b$. If $b = 1$ there is no problem. So assume that $b = 2$. Then in the earlier notation $M_1 \simeq V \otimes V^q$ where V is the usual module for $SL(2, q)$. We must show that this does not occur. Let $S_0 \in \mathrm{Syl}_2(N_1^{(\infty)})$ and let H_1 be a t-invariant 2-complement in $N_{N_1^{(\infty)}}(S_0)$. So H_1 is cyclic of order q^2-1. The action of H_1 on M_1 shows that $\bar{H}_1^{q+1}$ has nontrivial centralizers on M_1. Hence $C_{W_1}(\bar{H}_1^{q+1}) \neq 1$. However $\bar{H}_1^{q+1}$ is covered by $\overline{C_{\bar{H}_1}(t)}$ and $C_{\bar{H}_1}(t)$ acts in the same way on V_1 and on W_1/V_1. As $C_{\bar{H}_1}(t)$ is fixed-point-free on V_1, this is impossible.

To complete the proof of (3.1) it suffices by Gaschutz' result [9, 17.4] to show that the Sylow 2-subgroups of $N_1^{(\infty)}$ split over W_1. So consider the group $N_1^{(\infty)}$ and note that $O(N_1^{\infty}) = 1$. Let $S_0 \in \mathrm{Syl}_2(N_1^{(\infty)})$ be such that $V \leqslant S_0$ and S_0 is $\langle t \rangle$-invariant. Viewing W_1 as an $\mathbb{F}_{q^2}$-module let W_0 be the 1-space of W_1 stabilized by S_0 and let $L = C_{S_0}(W_1/W_0)$. Then $N(L) \cap N_1^{(\infty)} = LDH$ where $\bar{D} \simeq SL(n-2, q^2)$, $[\bar{D}, \bar{H}] = 1$, $Z(\bar{D}) \leqslant \bar{H}$ is cyclic of order q^2-1, and $C_L(\bar{H}) = 1$ or W_0, the latter occurring only if $\bar{H} = Z(\bar{D})$. We may take H to be cyclic of order q^2-1, so the Frattini argument implies that $N(L) \cap N_1^{(\infty)} \leqslant LN(H)$ and $L \cap N(H) = 1$ or W_0. Checking multipliers we see that $(N(H) \cap N_1)^{(\infty)}$ complements L in LD unless $\bar{D} \simeq SL(3, 4)$. So except for this one case, which we handle later, we may assume $DH \simeq \bar{D}\bar{H}$.

Recall the subgroups $V_1 \leqslant W_1$ and V_2 from Section 2. We have $V_2 \leqslant W_1 L$ and $V_2 \cap W_1 = V_1 \cap V_2 \leqslant W_0$. Observe that L/W_0 is elementary Abelian. Indeed L/W_0 contains W_1/W_0 as a normal subgroup and DH is transitive on L/W_1. Also W_1/W_0 is central in L/W_0 and $L - W_1$ contains involutions, as $V_2 \nleqslant W_1$. Therefore each coset of W_1/W_0 in L/W_0 contains involutions, and L/W_0 is elementary Abelian. Suppose $n = 3$. Then $q \geqslant 4$ and there is a subgroup $H_1 \leqslant N_1^{(\infty)}$ such that $H_1 \leqslant N(L)$ and $\bar{H}_1$ is cyclic of order q^2-1. Also $L = S_0$. Now H_1 acts on L/W_0, and L/W_0 is the sum of inequivalent $\mathbb{F}_2(H_1)$-submodules, $L/W_0 = W_1/W_0 \times C/W_0$. One checks that C/W_0 and W_0 are inequivalent as $\mathbb{F}_2(H_1)$-modules, so C is not homocyclic. As $|C/W_0|$ is a square,

C is not a Suzuki 2-group [3] and it follows that C is elementary Abelian. Then S_0 splits over W_1 as required. We now assume $n \geqslant 4$.

Assume $q > 2$. Then in $N_A(V_1)$ there is a nontrivial subgroup H_1 with $|H_1| \mid q-1$ and centralizing $L_1 \cong SL(n-1, q)$. We may choose $H_1 \leqslant N(W_1) = N_1$ and it follows that $\bar{H}_1 \leqslant C_{\bar{N}_1}(\bar{N}_1^{(\infty)})$. As H_1 is fixed-point-free on V_1, it is also fixed-point-free on W_1. An application of the Frattini argument now implies that $N_1 = W_1 N_{N_1}(O(N_1)\, H_1)$, a semidirect product. Again we have the result.

Finally assume $q = 2$. Then $n \geqslant 5$, $|V_2| = 2^{n-1}$, and $|V_2/V_2 \cap W_1| = 2^{n-2}$. For each involution $x \in V_2 - W_1$, $[L, x] = W_0$ and so $|C_L(x)| = \frac{1}{4}|L|$. Therefore $|C_L(V_2)| \geqslant (\frac{1}{4})^{n-2}|L| = (\frac{1}{4})^{n-2}\, 4^{2n-3} = 4^{n-1} = |W_2|$. Also $C_L(V_2) \cap W_1 = W_0$, so $L = W_1 C_L(V_2)$. The group $L_1 \cap L_2 \cong SL(n-2, q)$ normalizes $C_L(V_2)$ inducing contragredient representations on $V_1/V_1 \cap V_2$ and $V_2/V_1 \cap V_2$. As $W_2\langle t\rangle \in \mathrm{Syl}_2(C(V_2))$ and $|C_L(V_2)| = |W_2|$, we conclude that $L_1 \cap L_2$ acts on $C_L(V_2)/W_0$ as the sum of two representations equivalent to $V_2/V_1 \cap V_2$. Now let $J \leqslant L_1 \cap L_2$ be a Singer cycle. So J is cyclic of order $2^{n-2} - 1$. From the above facts it follows that L/W_0 decomposes as the sum of four irreducible $\mathbb{F}_2(J)$-submodules; W_1/W_0 is the sum of two equivalent $\mathbb{F}_2(J)$-submodules and $L/W_0 = W_1/W_0 \times X/W_0$, where $X/W_0 \geqslant V_2W_0/W_0$ and is the sum of two ireducible submodules contragredient (thus inequivalent) to those in W_1/W_0. Now $\bar{J} \leqslant \overline{L_1 \cap L_2} \leqslant \bar{D}$ so $\bar{J}$ is centralized by a Singer cycle in $\bar{D}$, having order $4^{n-2} - 1$ or $\frac{1}{3}(4^{n-2} - 1)$. This group then normalizes X as does $L_1 \cap L_2$. Therefore $D \leqslant N(X)$(see [14]) and since D is transitive on X/W_0, X must be elementary Abelian. As $|X| = |W_2|$, $X = W_2$. We now conclude that $E(N_2/W_2O(N_2)) \cong SL(n-1, q^2)$ and the action on W_2 is as on the usual module. This determines the action of D on X and so $X = W_0 \times X_1$, where X_1 is D-invariant. So a Sylow 2-subgroup of $X_1 D$ complements W_1 in a Sylow 2-subgroup of $N_1^{(\infty)}$, except in the one possible case of $\bar{D} \cong SL(3, 4)$.

In the exceptional case $\bar{H} \leqslant \bar{D}$. Suppose N_1 contains a subgroup inducing $GL(4, 4)$ on W_1. Then we easily see that $N_1^{(\infty)}$ splits over W_1, as desired. So we suppose this is not the case.

Let $S \in \mathrm{Syl}_2(W_1 W_2 D)$ with $V \leqslant S$. Choose $I \leqslant N_1^{(\infty)}$ elementary of order 3^3 and such that $\bar{I} \leqslant N(\bar{S})$ is the Cartan subgroup with respect to the root system Σ_1 spanned by $\{\pm\alpha_2, \pm\alpha_3, \pm\alpha_4\}$. Let $r \in \Sigma_1$ and $\bar{Z}_r$ the root subgroup of $\bar{N}_1^{(\infty)}$ containing $\bar{V}_r$. Let $I_0 = C_I(\bar{Z}_r)$. One checks that $I_0 \cong Z_3 \times Z_3$ and $C_{W_1}(I_0) = 1$. So if D_r is the preimage of Z_r we have $D_r = W_1 Z_r$ and $W_1 \cap Z_r = 1$, where $Z_r = C_{D_r}(I_0)$. Each Z_r is I-invariant. Also viewing W_1 as an $\mathbb{F}_4$-space we see that there are precisely four 1-spaces of W_1 invariant under I. Label these $Z_1 = Z_{\alpha_1}$, $Z_2 = Z_{\alpha_1+\alpha_2}$, $Z_3 = Z_{\alpha_1+\alpha_2+\alpha_3}$, and $Z_4 = Z_{\alpha_1+{}_2\alpha+\alpha_3+\alpha_4}$ in such a way that $1 < Z_4 < Z_4 \cdot Z_3 < Z_4 \cdot Z_3 \cdot Z_2 < W_1$ is an IS-composition series of W_1. We then have $S = \prod_{r\in\Sigma^+} Z_r$ and one checks that for $r \neq s$ in Σ^+, $C_I(Z_r) \neq C_I(Z_s)$. Now the proof of Lemma 3 in [22] shows that every I-invariant subgroup of S is a product of some of the root subgroups Z_r, $r \in \Sigma$.

The earlier arguments show that $I \leqslant N(W_2)$. It follows that $N_2^{(\infty)}/W_2 \simeq SL(4, 4)$ and as $[S, I] = S$, $S \in \mathrm{Syl}_2(N_2^{(\infty)})$. If N_2 contains a subgroup inducing $GL(4, 4)$ on W_2, then as above $N_2^{(\infty)}$ splits over W_2. It follows that $W_0 D$ splits over W_0. So assume this is not the case. Then $I \leqslant N_2^{(\infty)}$ and we carry through the above process of labeling root subgroups with respect to $N_2^{(\infty)}$. This labeling necessarily coincides with the earlier labeling since in each case the root subgroups in S are the minimal I-invariant subgroups of S.

Considering matrices in $N_1^{(\infty)}$ we see that $Z_{\alpha_2} \leqslant N(Z_{\alpha_1} Z_{\alpha_1+\alpha_2})$. So $Y = \langle Z_{\alpha_1}, Z_{\alpha_2}\rangle = Z_{\alpha_1} Z_{\alpha_2} Z_{\alpha_1+\alpha_2}$ and $Y \cap W_2 = 1$. It follows that $W_2 Y$ is N_2-conjugate to $W_2(S \cap D)$, so by Gaschutz' theorem D splits over W_2 and hence over L as needed.

(3.2) (i) *$E(\bar{N}_1)\langle \bar{w}, \bar{t}\rangle = E(\bar{N}_1)\langle \bar{w}\rangle$ and $E(\bar{N}_1) \cap \langle \bar{w}\rangle = 1$.*

(ii) *t induces a field automorphism on $E(\bar{N}_1)$.*

(iii) *a Sylow 2-subgroup S_1 of N_1 has the form $S_1 = S_0\langle w_1\rangle$, where $S_0 \in \mathrm{Syl}_2(N_1^{(\infty)})$ and $S_0 \cap \langle w_1\rangle = 1$. Also $t \in \langle w_1\rangle$ and we may take $V \leqslant S_0$.*

Proof. (ii) follows from the structure of $\mathrm{Aut}(L(n-1, q^2))$. By Lang's theorem [10] each involution in $E(\bar{N}_1)\langle \bar{t}\rangle$ is $E(\bar{N}_1)$-conjugate to $\bar{t}$. It follows from this and $t^g \cap tW_1 = t^{W_1}$, that $N_1 = N_1^{(\infty)} C(t)$. Considering $\mathrm{Aut}(L(n-1, q^2))$ we have (i). For (iii) first obtain an N_1-conjugate w_1 of w with $t \in \langle w_1\rangle$. Then choose $S_1 \in \mathrm{Syl}_2(N_1)$ with $V\langle w_1\rangle \leqslant S_1$ and set $S_0 = S_1 \cap N_1^{\infty}$.

(3.3) *Let $S_1 = S_0\langle w_1\rangle$, where $V \leqslant S_0 \in \mathrm{Syl}_2(N_1^{(\infty)})$ and $S_1 \in \mathrm{Syl}_2(N_1)$. There is an involution $d \in N(S_1) - S_1$ such that $S = S_1\langle d\rangle \in \mathrm{Syl}_2(G)$ and $d \in C(t)$.*

Proof. First we show that $S_1 \notin \mathrm{Syl}_2(G)$. For suppose otherwise. If $t^G \cap S_0 = \varnothing$, then transfer applies and we have $t \notin G'$. Then $S_1 \cap G' = S_0$ (as $S_1 \leqslant N_1^{(\infty)}$), so G' has Sylow 2-subgroups of type $L_n(q^2)$. By McBride [12], $E(G'/O(G')) \simeq L_n(q^2)$. So by (1.3), $E(G')/Z(G') \simeq L_n(q^2)$, whereas G is assumed to be a counterexample to the main theorem. Therefore $t^G \cap S_0 \neq \varnothing$. By (2.1), $t^G \cap U = \varnothing$ and since $W_1^{\#}$ is fused in N_1, $t^G \cap W_1 = \varnothing$. Say $t^g \in S_0 - W_1$ and t^g has precisely l nontrivial Jordan blocks on W_1, when W_1 is viewed as an $\mathbb{F}_{q^2}$-module. Write $S_0 = W_1 S_{00}$ where $W_1 \cap S_{00} = 1$. We may assume that S_0 normalizes the l-space $[W_1, t^g]$ and that a basis has been chosen for W_1 in such a way that S_{00} is represented on W_1 by lower triangular matrices and t^g is represented by the matrix

$$\begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & \ddots & & & \\ 1 & & & 1 & & \\ & \ddots & & & \ddots & \\ & & \ddots & & & \ddots \\ & & & 1 & & & 1 \end{bmatrix}.$$

Let $t^g = sw$ with s an involution in S_{00} and $w \in W_1$. Then $w \in C_{W_1}(t^g) = C_{W_1}(s)$, which has dimension $n - 1 - l$. Set $J = \langle s^{S_{00}} \rangle$. Then $C_{W_1}(s) = C_{W_1}(J)$. Moreover $C_{W_1}(t^g)$ and $C_{W_1}(t^g)J$ are normal in S_0, so $[S_0, t^g] \leqslant C_{W_1}(t^g)[J, S_0]$.

One checks that $[J, S_0] \leqslant C_{W_1}(t^g)[J, S_{00}]$ and $|[J, S_{00}]| = (q^2)^{(1/2)l(l-1)} = q^{l(l-1)}$. Therefore $|[S_0, t^g]| \leqslant q^{l(l-1)}(q^2)^{n-l-1}$ and $|C_{S_0}(t^g)| \geqslant q^a$ for $a \geqslant n(n-1) - (l(l-1) + 2(n-l-1))$. If $n \geqslant 4$, then $|C_{S_0}(t^g)|$ has order strictly greater than the order of a Sylow 2-subgroup of $C(t)$. This is a contradiction. If $n = 3$, $C_{S_0}(t^g)$ contains an elementary subgroup of order q^4. Again this is impossible. So we must have $S_1 \notin \mathrm{Syl}_2(G)$ as claimed.

Choose $d \in N(S_1) - S_1$ with $d^2 \in S_1$. As above we have $t^G \cap S_0 = \varnothing$, so $t^d \in tS_0$. As t induces a field automorphism on S_0, $t^d = t^s$ for $s \in S_0$, and we may take $d \in C(t)$. Then $V\langle w, d\rangle \in \mathrm{Syl}_2(N(A))$. The structure of $\mathrm{Aut}(S_0)$ is known [11] and from the description there it is clear that $t \notin (S_0\langle w, d\rangle)'$. So either we may take d as an involution or $w = t = d^2$. Suppose the latter occurs. If $S_0\langle d\rangle \in \mathrm{Syl}_2(G)$, then by the above argument $t^G \cap S_0 = \varnothing$. But then we use transfer and [12] to get a contradiction. If $S_0\langle d\rangle \notin \mathrm{Syl}_2(G)$, then as above there exists a 2-element $g \in C(t) \cap N(S_0\langle d\rangle)$ and $g \notin S_0\langle d\rangle$. This is impossible as $V\langle w, d\rangle = V\langle d\rangle \in \mathrm{Syl}_2(N(A))$. We may now assume $d^2 = 1$.

Finally we must show that $S_1\langle d\rangle \in \mathrm{Syl}_2(G)$. So suppose not and let $s \in N(S_1\langle d\rangle) - S_1\langle d\rangle$. We have seen that $t^G \cap S_0 = \varnothing$, so $t^s \in tS_0$, dS_0, or dtS_0. Each factor in the lower central series for S_0 is a free $\mathbb{F}_2\langle t\rangle$-module, so it follows from (1.2) that each involution in S_0t is conjugate in S_0 to t. As S_1 contains $V\langle w, d\rangle \in \mathrm{Syl}_2(C(t))$ we conclude $t^s \notin tS_0$. Replacing d by dt, if necessary, we may assume $t^s \in dS_0$. Also we see that s normalizes S_0 as follows. Note that $Z = Z(S_0) = C(S_1\langle d\rangle)' \cap S_1\langle d\rangle$, so $s \in N(Z(S_0))$. Next $Z_2(S_0) = Z_2$ where $Z_2/Z_1 = C((S_1\langle d\rangle)'Z_1/Z_1)$. Continuing we get $S_0 = Z_{n-1}$ to be s-invariant. It follows that each involution in dS_0 is S_0-conjugate to d and we may assume that $t^s = d$.

Now d interchanges V_1, V_2 and $s \in N(S_0)$. Choosing $W_1, W_2 \leqslant S_0$ we have d interchanging W_1 and W_2. From the structure of $\mathrm{Aut}(S_0)$ [11] we have a contradiction. So $S_1\langle d\rangle \in \mathrm{Syl}_2(G)$ and the proof of (3.3) is complete.

(3.4) *G contains a normal subgroup G_0 such that $S_0\langle d_1\rangle \in \mathrm{Syl}_2(G_0)$, where $d_1 = d$ or dt.*

Proof. Consider the transfer into $S_1\langle d\rangle/S_0\langle d\rangle$. To transfer out $\langle w\rangle$ it suffices to show that $t^G \cap S_0\langle d\rangle = \varnothing$. Once this is accomplished, we have the result by using the fact that $S_0 \leqslant N(S_0)' \leqslant G'$.

Suppose then that $t^G \cap S_0\langle d\rangle \neq \varnothing$. As in (3.3) $t^G \cap S_0 = \varnothing$, so suppose $t^g \in S_0d$. Consider t^g acting on $X = (N_1 \cap N_2)'$ and assume $n \geqslant 5$. Then X induces $SL(n-2, q^2)$ on $W_1W_2/W_1 \cap W_2$. Replacing d by dt, if necessary, we may assume that d induces a graph-field automorphism on $X/O_2(X)$, so by [2, (19.6)] we have t conjugate to an involution in dW_1W_2. So take $t^g \in dW_1W_2$, and since $W_1W_2/W_1 \cap W_2$ is a free $\mathbb{F}_2\langle d\rangle$-module, we may take $t^g \in d(W_1 \cap W_2)$. Since $X \leqslant C(W_1 \cap W_2)$ and $W_1W_2/(W_1 \cap W_2)$ is a free $\mathbb{F}_2\langle d\rangle$-module we have $C(t^g)$ covering $O^2(C(d) \cap (X/W_1W_2))$. But this implies that $E(C(t)) \sim E(C(t^g))$ contains a section isomorphic to $O^2(SU(n-2, q))$, which is impossible.

So $n = 3$ or 4. First assume that $n = 4$. Then $X = (N_1 \cap N_2)'$ induces $SL(2, q^2)$ on $W_1W_2/W_1 \cap W_2$ and d or dt must centralize $X/O_2(X)$. Replacing d by dt, if necessary, we may assume $t^g \in dS_0$ and $[t^g, S_0]$, $[d, S_0] \leqslant W_1W_2$. As t^g interchanges W_1 and W_2, the stabilizer in X of $t^g(W_1 \cap W_2)$ covers $X/O_2(X)$ and has order $q^4 \mid X/O_2(X)|$. From the action of X on $O_2(X)$ we conclude that $C_X(t^g)$ covers $X/W_1 \cap W_2$. But $C_G(t)$ contains no subgroup isomorphic to $C_X(t^g)$. This is impossible. Therefore $n = 3$. Choose a subgroup $H \leqslant A$ with H d-invariant, H cyclic of order $q - 1$, H fixed-point-free on $V_1 \cap V_2$. Then H is fixed-point-free on $W_1 \cap W_2$. Also $[H, d] = 1$ or $[H, d]$ has order $q^{1/2} + 1$, depending on whether $\widetilde{C_A(d)} \cong SL(2, q)$ or $U_3(q^{1/2})$. Considering $H\langle d\rangle$ acting on $W_1 \cap W_2$ we conclude that $C(d) \cap W_1 \cap W_2$ has order q or q^2. In the first case stabilizer in S_0 of $t^g(W_1 \cap W_2)$ has order q^4 and since $t^G \cap t^g(W_1 \cap W_2) = t^g(C_{W_1 \cap W_2}(t^g))$, $|C_{S_0}(t^g)| = q^3$. But as $C_{S_0}(t^g)$ contains no involution outside $W_1 \cap W_2$, this is impossible. In the second case $W_1 \cap W_2 \sim V_i$, $i = 1$ or 2, and $C(V_i)$ contains a subgroup conjugate to $C(W_1 \cap W_2) \geqslant S_0$. This is impossible.

(3.5) $A/Z(A) \not\cong L_3(q)$.

Proof. Let G_0 be as in (3.4). If $d_1^{G_0} \cap S_0 = \varnothing$, then by transfer $S_0 \in \mathrm{Syl}_2(\langle S_0{}^G\rangle)$. McBride's result [12] gives a contradiction. Therefore $d_1{}^g \in S_0$ for some $g \in G_0$. As A contains just one class of involutions, $C_G(d_1)$ contains a conjugate $S_0{}^y$ of S_0 on which t acts. However d_1 acts on A with $C_A(d_1) \cong L_2(q)$ or $U_3(q^{1/2})$. In either case we have a contradiction by noting that $C_A(d_1)$ is standard in $C_G(d_1)$ and applying [6].

(3.6) $\bar{A} \not\cong L_n(q)$, $n \geqslant 4$.

Proof. As in (3.5) we may assume $d_1{}^G \cap S_0 \neq \varnothing$. Conjugating by an element of $\langle N_1, A\rangle$ we may choose $d_1{}^g \in S_0 - W_1$. Say $d_1{}^g$ has precisely l nontrivial Jordan blocks in its action on W_1, and choose a basis for W_1 so that S_0/W_1 is represented by lower triangular matrices. Then we may assume that $d_1{}^g$ induces the matrix

$$\begin{bmatrix} 1 & & & & \\ & \cdot & & & \\ 1 & & \cdot & & \\ & \ddots & & \cdot & \\ & & \ddots & & \\ & & 1 & & 1 \end{bmatrix}$$

on W_1, and we may write $d_1{}^g = w_1v$ where v is an involution in V. Now W_1

has the form $W_1 = W_{11} \times W_{12}$, where v centralizes W_{11}, $|W_{12}| = (q^2)^{2l}$, and v induces l Jordan blocks on W_{12}. By (1.2) we may assume that $w_1 \in W_{11}$. Also we may choose W_{11} and W_{12} so that $W_{11} \cap V_1 \neq 1$. Then $C_L(vW_1)$ contains a subgroup isomorphic to $SL(n-2l, q)$ that centralizes W_{12} and is transitive on $W_{11}^{\#}$. Therefore we may assume that $w_1 \in V_1 \cap W_{11}$, so that $d_1^g \in V \leqslant A$. Now choose k minimal and a conjugate $d_1^{g^{-1}}$ of d_1 such that $d_1^{g^{-1}} \in A$ and $d_1^{g^{-1}}$ has precisely k nontrivial Jordan blocks on the usual $\mathbb{F}_q$-module for $SL(n, q)$.

The above arguments show that each involution in S_0 is conjugate to one in A. Take $d_1^g \in V$ and view $S_0 \in \mathrm{Syl}_2(SL(n, q^2))$. We may assume that d_1^g is in Suzuki form ([2], §4). Let $y \in S_0$ be any involution of Jordan rank r and in Suzuki form. Let M be the natural module for $SL(n, q^2)$ and set $J_r = C_{S_0}(C_M(y)) \cap C_{S_0}(M/[M, y])$. Then $J_r \lhd S_0$, $C_M(J_r) = C_M(y)$, and $|J_r| = (q^2)^{r^2}$.

Choose $(S_0\langle d_1\rangle)^h$ t-invariant with $d_1 \in J_k{}^h \leqslant S_0{}^h$. As in (3.3) $S_0{}^h$ is characteristic in $S_0{}^h\langle d_1{}^h, t\rangle$. Then t acts on $S_0{}^h$ and on a normal series of $S_0{}^h$ with each factor a free $\mathbb{F}_2\langle t\rangle$-module. Consequently each involution in $S_0{}^h t$ is conjugate to t. In particular $td_1 \sim t$.

Suppose n is odd. Let $X = (N_1 \cap N_2)'$, so X induces $SL(n-2, q^2)$ on $W_1W_2/(W_1 \cap W_2)$. By [2, (19.6)], each involution in Xtd_1 is conjugate to an involution in $W_1W_2td_1$. Arguing as at the end of the second paragraph in the proof of (3.4), we have $E(C(td_1)) \sim E(C(t))$ containing a section isomorphic to $U_{n-2}(q)$ or $Sp(n-3, q^2)$. This is impossible.

Say n is even, $n = 2b$. Then J_b is the unique maximal elementary subgroup of S_0 of order q^{2b}. Also $J_k \leqslant J_b$. So t normalizes $J_b{}^h$, centralizing a subgroup of order at least q^b. Thus $C(t) \cap C(d_1)$ contains an elementary subgroup of order at least q^b. This is a contradiction and proves (3.6).

4. $\widetilde{E(\overline{N}_i)} \cong L_{n-1}(q) \times L_{n-1}(q)$ FOR $i = 1$ AND 2

In this section we complete the proof of the main theorem by considering the case of (2.7) not treated in Section 3. That is, we assume that $\widetilde{E(\overline{N}_i)} \cong L_{n-1}(q) \times L_{n-1}(q)$ for $i = 1$ and 2. Recall that G is a minimal counterexample to the main theorem.

We write $\widetilde{E(\overline{N}_i)} = \widetilde{\overline{L}_{i1}} \times \widetilde{\overline{L}_{i2}}$, where $\overline{L}_{i1}$ and $\overline{L}_{i2}$ are quasi-simple subgroups interchanged by t.

The first result shows that the module W_i decomposes under the action of $\overline{L}_{i1}\overline{L}_{i2}$.

(4.1) $W_i = W_{i1} \times W_{i2}$ *with* $\overline{L}_{ij} \cong SL(n-1, q)$ *acting in the usual way on* W_{ij} *(when viewed as an* $\mathbb{F}_2$*-module) and centralizing* W_{ik} *for* $k \neq j$.

Proof. We first claim that W_i when restricted to $\bar{L}_{i1} \times \bar{L}_{i2}$ is reducible. Suppose this is not the case. Then $W_i \mid \bar{L}_{i1}$ is homogeneous. If $W_i \mid \bar{L}_{i1}$ were irreducible, then as $\bar{L}_{i2} \leqslant C(\bar{L}_{i1})$, Schur's lemma implies that $\bar{L}_{i2}$ acts as a cyclic group on W_i ; impossible. Therefore $W_i \mid \bar{L}_{i1}$ is reducible. If $\bar{L}_{ij}$ acts nontrivially on an elementary Abelian 2-group of order 2^a, then $2^a \geqslant q^{n-1}$. Indeed this follows from [3, (4B)] if $n = 3$ and was proved in (3.1) if $n \geqslant 4$. So $W_i \mid \bar{L}_{i1}$ is the sum of two $\bar{L}_{i1}$-invariant submodules Z_1, Z_2 on which $\bar{L}_{i1}$ induces equivalent representations.

Passing to the $\mathbb{F}_q$-module $\hat{W}_i = W_i \otimes \mathbb{F}_q$ we have $\hat{W}_i = M_1 \oplus \cdots \oplus M_r$, where the M_i are absolutely irreducible, algebraically conjugate, $\bar{L}_{i1}\bar{L}_{i2}$-modules over $\mathbb{F}_q$. So each $M_m = M_{m1} \otimes M_{m2}$ where M_{mj} is an (absolutely) irreducible $\mathbb{F}_q$-module for $\bar{L}_{ij}$. As an $\mathbb{F}_q(\bar{L}_{i1})$-module $\hat{W}_i \cong (Z_1 \otimes \mathbb{F}_q) \oplus (Z_2 \otimes \mathbb{F}_q)$ and $Z_1 \otimes \mathbb{F}_q \cong Z_2 \otimes \mathbb{F}_q$ is the sum of inequivalent, absolutely irreducible, modules for $\bar{L}_{i1}$. But $M_m \mid \bar{L}_{i1}$ is the sum of $\dim_{\mathbb{F}_q}(M_{m2})$ equivalent $\mathbb{F}$-modules for $\bar{L}_{i1}$. So as $\dim(M_{m2}) \geqslant n - 1$, we have $n - 1 = 2$ and $\bar{L}_{i1} \cong SL(2, q)$.

It is easy to see that $\bar{L}_{i1}$ stabilizes precisely $q + 1$ subgroups of W_i of order q^2. So a Sylow 2-normalizer of $\bar{L}_{i2}$, say $\bar{J}_{i2}$, stabilizes one such subgroup I of W_i. As $\bar{L}_{i1} \times \bar{J}_{i2}$ acts irreducibly on I, $O_2(\bar{J}_{i2})$ centralizes I. By symmetry the Sylow 2-subgroups of $\bar{L}_{i1}$ centralize a subgroup of W_i of order at least q^2. It follows from [3(4B), (4C)] that L_{i1} acts on I as on the usual module for $\bar{L}_{i1} \cong SL(2, q)$ when viewed as an $\mathbb{F}_2$-module. From here it follows that each M_{mj} is an algebraic conjugate of the usual $\mathbb{F}_q$-module for $\bar{L}_{ij}$. However, we know also that L_i acts on W_i/V_i and on V_i as on the usual module for $SL(2, q)$, viewed as an $\mathbb{F}_2$-module. So if we let h be an element of order $q - 1$ in L_i, then there is an element $\alpha \in \mathbb{F}_q^{\#}$ such that the eigenvectors of h on $\hat{W}_i$ are the algebraic conjugates of α and α^{-1}, each with multiplicity two. This is not consistent with the form of $M_1, \ldots, M_r$. Therefore W_i is reducible when restricted to $\bar{L}_{i1}\bar{L}_{i2}$, as claimed.

Let W_{i1} be an irreducible submodule of W_i under $\bar{L}_{i1}\bar{L}_{i2}$, and choose notation so that $\bar{L}_{i1}$ is not trivial on W_{i1}. Then as before $\mid W_{i1} \mid \geqslant q^{n-1}$, and since W_i/W_{i1} is also nontrivial $\mid W_{i1} \mid = \mid W_i/W_{i1} \mid = q^{n-1}$. Moreover $\bar{L}_{i1}$ acts irreducibly on W_{i1}. Then $\bar{L}_{i2} \leqslant C(\bar{L}_{i1})$ forces $\bar{L}_{i2}$ to be trivial on W_{i1}. As t interchanges $\bar{L}_{i1}$ and $\bar{L}_{i2}$ we set $W_{i2} = W_{i1}^t$ and get $W_i = W_{i1} \times W_{i2}$.

It remains to show that as an $\mathbb{F}_2$-module, W_{ij} is equivalent to the usual module for $\bar{L}_{ij}$. But this follows from the fact that L_i acts on W_i/V_i and on V_i as on the usual module. The proof of (4.1) is complete.

(4.2) *Notation.* Let $S_0{}^t \in \mathrm{Syl}_2(N_1^{(\infty)})$ with $C_{S_0}(t) = V$. By (4.1) we have $S_0 = A_1 \times B_1$ with $B_1 = A_1{}^t$. Here $A_1 \geqslant W_{11}$ and $B_1 \geqslant W_{12}$. As $V \in \mathrm{Syl}_2(L_n(q))$ this is also true of A_1 and B_1.

Recall the (B, N)-notation for A. We have a root system Σ and $V = \prod_{r \in \Sigma^+} U_r = \langle U_{\alpha_1}, \ldots, U_{\alpha_{n-1}} \rangle$. Accordingly we write

$$A_1 = \prod_{r \in \Sigma^+} Y_r, \ B_1 = \prod_{r \in \Sigma^+} Z_r, \text{ where } U_r \leqslant Y_r \times Z_r \text{ and } Z_r = Y_r{}^t.$$

Let the Weyl group of A be generated by reflections $s_1,\ldots, s_{n-1}$ with $s_i \in \langle U_{\alpha_i}, U_{-\alpha_i}\rangle$ and $U_{-\alpha_i} = U_{\alpha_i}^{s_i}$.

(4.3) *There exist subgroups G_1, G_2 of G with the following properties*:

(i) $G_1^t = G_2$.

(ii) $[G_1, G_2] = 1$.

(iii) $G_1 \cap G_2 \leqslant Z(G_1G_2)$ *and* $|Z(G_1G_2)|$ *is odd.*

(iv) $G_1G_2/Z(G_1G_2) \cong L_n(q) \times L_n(q)$.

Proof. First note that in the group $N_2^{(\infty)}$ we have subgroups corresponding to those in (4.2) which we label $\hat{A}_1, \hat{B}_1, \hat{Y}_r, \hat{W}_{21}, \hat{W}_{22}$. Then $V = C(t) \cap (\hat{A}_1 \times \hat{B}_1)$. We know that $C(V_2) \cap A_1B_1$ is elementary of order $(q^{n-1})^2 = |W_2|$, so we may take $W_2 \leqslant A_1B_1$. It then follows that $W_2 \lhd A_1B_1$ and conjugating if necessary we may assume $A_1B_1 = \hat{A}_1\hat{B}_1 = S_0$.

We have $N_1^{(\infty)} = M_{11} \times M_{12}$ where $W_{1i} = O_2(M_{1i})$ and $\overline{M}_{1i} = \bar{L}_{1i}$. As t interchanges M_{11} and M_{12}, and $C_{N_1^{(\infty)}}(t) = V_1L_1$, we may write $M_{1i} = W_{1i}L_{1i}$, where $L_{1i} \cong SL(n-1, q)$, $L_{12} = L_{11}^t$, and $L_1 = C(t) \cap L_{11}L_{12}$. We may assume that $W_{11} = Y_{\alpha_1} \cdots Y_{\alpha_1+\cdots+\alpha_{n-1}}$ and $A_1 \cap L_{11} = \langle Y_{\alpha_2},\ldots, Y_{\alpha_{n-1}}\rangle$. Then $B_1 \cap L_{12} = \langle Z_{\alpha_2},\ldots, Z_{\alpha_{n-1}}\rangle$. Also $L_{11} = \langle Y_{\pm\alpha_2},\ldots, Y_{\pm\alpha_{n-1}}\rangle$, where for $i = 2,\ldots, n-1$ we write $Y_{-\alpha_i} = Y_{\alpha_i}^{s_i}$. Similarly $L_{12} = \langle Z_{\pm\alpha_2},\ldots, Z_{\pm\alpha_{n-1}}\rangle$.

The Krull–Schmidt theorem implies that with a possible change of notation $\hat{A}_1 \leqslant A_1Z(S_0)$ and $\hat{B}_1 \leqslant B_1Z(S_0)$. We claim that $A_1 = \hat{A}_1$ and $B_1 = \hat{B}_1$. Suppose that $n = 3$. Then $q \geqslant 4$ and $N_1 \cap N(W_1W_2)$ contains an Abelian subgroup X_1X_2 such that $L_{1i} \geqslant X_i$ is cyclic of order $q-1$ and fixed-point-free on $S_0 \cap M_{1i}$. Viewing this in N_2 we conclude that $A_1 = \hat{A}_1$ and $B_1 = \hat{B}_1$.

Now suppose $n \geqslant 4$. Then L_{11} contains a subgroup $X_1 \cong SL(n-2, q)$ normalizing W_2. Namely, $X_1 = \langle Y_{\pm\alpha_2},\ldots, Y_{\pm\alpha_{n-2}}\rangle$. Then $[X_1, W_1W_2] = W_1W_2 \cap M_{11}$. Considering $X_1 \times X_1^t \leqslant N_2^{(\infty)}$ we see that X_1 induces a subgroup of $\bar{L}_{21}$ on W_1W_2. Therefore $[X_1, W_1W_2] \leqslant M_{21}$. Therefore $W_1W_2 \cap M_{11} \leqslant \hat{A}_1$. Since $A_1 \cap X_1 \leqslant \hat{A}_1$ we have $A_1 = (A_1 \cap X_1)(W_1W_2 \cap M_{11}) \leqslant \hat{A}_1$, so by orders $A_1 = \hat{A}_1$. Similarly $B_1 = \hat{B}_1$.

Now write $N_2^{(\infty)} = \hat{M}_{21} \times \hat{M}_{22}$, where $\hat{M}_{2i} = \hat{W}_{2i}\hat{L}_{2i}$. Then $\hat{W}_{21} = Y_{\alpha_{n-1}} \cdots Y_{\alpha_1+\cdots+\alpha_{n-1}}$, $\hat{W}_{22} = Z_{\alpha_{n-1}} \cdots Z_{\alpha_1+\cdots+\alpha_{n-1}}$, $A_1 \in \mathrm{Syl}_2(\hat{M}_{21})$, and $B_1 \in \mathrm{Syl}_2(\hat{M}_{22})$. We may also choose $\hat{L}_{21}, \hat{L}_{22}$ to be interchanged by t and $\langle U_{\pm\alpha_1},\ldots, U_{\pm\alpha_{n-2}}\rangle = L_2 = C(t) \cap (L_{21}L_{22})$. Then $\hat{L}_{21}$ can be expressed as $\hat{L}_{21} = \langle \hat{Y}_{\pm\alpha_1},\ldots, \hat{Y}_{\pm\alpha_{n-2}}\rangle$ and $\hat{L}_{22} = \langle \hat{Z}_{\pm\alpha_1},\ldots, \hat{Z}_{\pm\alpha_{n-2}}\rangle$. We know that $A_1 \leqslant \hat{M}_{21}$ and $C_G(t) \cap \hat{M}_{21} = 1$. Considering the map $g \to [g, t]$ for $g \in M_{21}$, we get $\hat{Y}_{\alpha_i} = Y_{\alpha_i}$ for $i = 1,\ldots, n-2$.

Let $w_0 \in \langle s_1,\ldots, s_{n-1}\rangle$ be the word of greatest length in the generators $s_1,\ldots, s_{n-1}$. Fix $i \in \{2,\ldots, n-1\}$. Then $\alpha_i^{w_0} = -\alpha_{n-i}$; w_0 can be expressed as $w_0 = s_is_{j_1} \cdots s_{j_r}$, where $r + 1 = l(w_0)$, and for $1 \leqslant k \leqslant r$, $(\alpha_i)\, s_{j_1},\ldots, s_{j_k} =$

$(-\alpha_i)\, s_i s_{j_1}, \ldots, s_{j_k} \in \Sigma^+$. As each $j_k \in L_1$ or L_2, we conclude that $(Y_{-\alpha_i})^{w_0} = (Y_{\alpha_i})^{s_{j_1} \cdots s_{j_r}} = Y_{\alpha_{n-i}}$. Also for $i = \{1, \ldots, n-2\}$ we have

$$(\hat{Y}_{-\alpha_i})^{w_0} = (\hat{Y}_{-\alpha_i})^{s_i s_{j_1} \cdots s_{j_r}} = (Y_{\alpha_i})^{s_{j_1} \cdots s_{j_r}} = Y_{\alpha_{n-i}}.$$

If $i \geqslant 2$, then $(\hat{Y}_{-\alpha_i})^{w_0} = (Y_{-\alpha_i})^{w_0}$ so $Y_{-\alpha_i} = \hat{Y}_{-\alpha_i}$. Now set $Y_{-\alpha_1} = \hat{Y}_{-\alpha_1}$. Conjugating these equations by w_0 and making similar arguments for the root groups $Z_{\pm\alpha_i}$, we have

$$(Y_{\pm\alpha_i})^{w_0} = Y_{\mp\alpha_{n-i}} \qquad \text{and} \qquad (Z_{\pm\alpha_i})^{w_0} = Z_{\mp\alpha_{n-i}} \quad \text{for } i = \{1, \ldots, n-1\}.$$

Notice that $G_1 = \langle L_{11}, L_{21} \rangle = \langle Y_{\pm\alpha_1}, \ldots, Y_{\pm\alpha_{n-1}} \rangle \leqslant C(B_1)$. By the above, w_0 normalizes $\langle L_{11}, L_{21} \rangle$ and $\langle B_1, B_1^{w_0} \rangle = \langle Z_{\pm\alpha_1}, \ldots, Z_{\pm\alpha_{n-1}} \rangle = G_2$. Therefore $[G_1, G_2] = 1$ and $G_2 = G_1{}^t$, proving the first two parts of (4.3).

Clearly $G_1 \cap G_2 \leqslant Z(G_1 G_2) = Z$ and $A \leqslant G_1 G_2$. As $Z(G_1 G_2) \leqslant \widetilde{C_G}(A)$, $|Z(G_1 G_2)|$ is odd, proving (iii). This also implies that $C(t)$ covers $C(t) \cap G_1 G_2 = C(t) \cap (G_1 Z/Z) \times (G_2 Z/Z)$. G_1 is generated by the perfect groups L_{11} and L_{21}, so G_1, and therefore G_2, is perfect. Consequently $G_1 G_2 / Z(G_1 G_2) \cong A/Z(A) \times A/Z(A) \cong L_n(q) \times L_n(q)$, completing the proof of (4.3).

All that remains is to prove that $G_0 = G_1 G_2 \trianglelefteq G$. For this set $\Omega = \{G_0{}^g : g \in G\}$.

(4.4) *t fixes a unique point of Ω, so $C(t) \leqslant N(G_0)$.*

Proof. Suppose t fixes $G_0{}^g$. If $t \in N(G_1{}^g) \cap N(G_2{}^g)$, then checking $C_{G_0{}^g}(t)$ we get a contradiction. So $(G_1{}^g)^t = G_2{}^g$ and hence $E(C_{G_0{}^g}(t))$ involves $L_n(q)$. We conclude that $A \leqslant G_0{}^g$. Arguing as in the proof of (2.7) we see that $N_G(V_i)^\infty$ centralizes $O(N_G(V_i))$ and hence $W_i = O_2(N_G(V_i))$. So W_i is the unique Sylow 2-subgroup of $C_G(V_i)$. As $(C_{G_0{}^g}(V_i))^{(\infty)}$ has the form $D_i L_i$, where D_i is normal and elementary of order $|V_i|^2$, $D_i = W_i$. Also $G_0{}^g = \langle N_{G_0{}^g}(D_1)^{(\infty)}, N_{G_0{}^g}(D_2)^{(\infty)} \rangle = \langle N_{G_0{}^g}(W_1)^{(\infty)}, N_{G_0{}^g}(W_2)^{(\infty)} \rangle$ and $N_{G_0{}^g}(W_i)^{(\infty)} \cong N_G(W_i)^{(\infty)}$. But $G_0 = \langle N(W_1)^{(\infty)}, N(W_2)^{(\infty)} \rangle$, so $G_0 = G_0{}^g$, proving the result.

(4.5) *G contains a normal subgroup G_{00} with $G = G_{00}\langle t \rangle$ and $t \notin G_{00}$.*

Proof. Let $X = N_G(G_1) \cap N_G(G_2) \leqslant N_G(G_0)$. Then $X\langle t \rangle$ contains a Sylow 2-subgroup of G. By the Thompson transfer lemma we need only show that $t^G \cap X = \varnothing$. Suppose $t^g \in X$. Then $t \in X^{g^{-1}} \leqslant N_G(G_0^{g^{-1}})$, so by (4.4), $g^{-1} \in N(G_0)$. But as $X \trianglelefteq N_G(G_0)$ this is impossible.

(4.6) *Let $g \in \mathrm{Aut}(S_1)$ have odd order, where $S_1 \in \mathrm{Syl}_2(G_1)$. Then g normalizes W_{11} and W_{21}.*

Proof. We have $S_1 \in \mathrm{Syl}_2(L_n(q))$, so if $q \geqslant 4$ the result follows from [11]. Suppose $q = 2$. Let $X = W_{11}$ and $Y = W_{11}^g$. Then X, and hence Y, has the property that $(X \cap Z_i(S_1))/(X \cap Z_{i-1}(S_1))$ has order 2 for each $i = 1, \ldots, n-1$, where $Z_i(S_1)$ is the ith term of the upper central series of S_1. As g normalizes $Z(S_1)$, $X \cap Y \geqslant Z(S_1)$. Choose i maximal with $X \cap Z_i(S_1) = Y \cap Z_i(S_1)$.

First suppose $i \geqslant 2$. If $i = n - 1$, then $X = Y$ and we have g normalizing W_{11}. Each factor $Z_{i+1}(S_1)/Z_i(S_1)$ is the direct product of images of certain root subgroups of S_1. For example $Z_{n-2}(S_1)/Z_{n-3}(S_1) \cong Y_{\alpha_1+\alpha_2} \times Y_{\alpha_2+\alpha_3} \times \cdots \times Y_{\alpha_{n-2}+\alpha_{n-1}}$. Using the commutator relations and the fact that $[Y \cap Z_{i+1}(S_1), S_1] \leqslant Y \cap Z_i(S_1) = X \cap Z_i(S_1)$ we see that $(Y \cap Z_{i+1}(S_1))\, Z_i(S_1) = (X \cap Z_{i+1}(S_1))\, Z_i(S_1)$. Let $y \in Y \cap (Z_{i+1}(S_1) - Z_i(S_1))$ and write $y = xv$ where $x \in X \cap Z_{i+1}(S_1)$ and $v \in Z_i(S_1)$. If $v \notin (X \cap Z_i(S_1))\, Z_2(S_1)$, then this leads to a contradiction. This is because it is then possible to choose an element $s \in S_1$ such that $[x, s] \in X \cap Z_i(S_1)$ and $[v, s] \notin X \cap Z_i(S_1)$. So $v \in (X \cap Z_i(S_1))Z_2(S_1)$. Now g normalizes each $X \cap Z_j(S_1) = Y \cap Z_j(S_1)$ for $j \leqslant i$, and also g normalizes $Z_2(S_1)$ which has order 8. It follows that g stabilizes a series of $(X \cap Z_{i+1}(S_1))\, Z_2(S_1)$ each factor of which has order 2. Since $|g|$ is odd, g must centralize $(X \cap Z_{i+1}(S_1))\, Z_2(S_1)$. In particular g centralizes $X \cap Z_{i+1}(S_1)$, which is not the case.

Therefore $i = 1$. In particular this implies $X \cap Y = Z(S_1)$, so $[X, Y] \leqslant Z(S_1)$ and $Y \leqslant C(X/Z(S_1)) = W_{11}W_{21}$. Then $W_{11}W_{21} = XY = C_{S_1}(X/Z(S_1)) = C_{S_1}(Y/Z(S_1))$ is g-invariant. Now S_1/XY is isomorphic to the Sylow 2-subgroups of $SL(n-2, 2)$. We may apply induction unless $n - 2 = 2$, 3, or 4. Note that $n - 2 = 2$ does not occur as $\tilde{A} \ncong L_4(2)$. Assume for the moment that $n - 2 > 4$.

Inductively g normalizes subgroups H/XY, K/XY, with H/XY, K/XY analogous to W_{11}, W_{21}, respectively. Relabeling, if necessary, we have the centralizer in $W_{i1}Z(XY)/Z(XY)$ of $H/Z(XY)$, respectively, a 1-space or hyperplane of $W_{i1}Z(XY)/Z(XY)$. For $K/Z(XY)$ the centralizers are, respectively, a hyperplane or 1-space of $W_{i1}Z(XY)/Z(XY)$. Now g normalizes $C(K/Z(XY)) \cap XY/Z(XY)$. This forces $C(H/XY) \cap X/Z(XY) \cap X^g/Z(XY) \neq 1$. However $X \cap X^g = X \cap Y = Z(XY)$, a contradiction.

We still have the cases $n = 5, 6$. If $n = 5$ we have $S/XY \cong D_8$ so g centralizes S/XY and the above argument works. Let $n = 6$. If g normalizes H/XY, K/XY we argue as before. Suppose g does not normalize H/XY, K/XY. Argue as in the second paragraph of the proof to conclude that $H/XY \cap K/XY = Z(S_1/XY)$. Then $\langle g \rangle$ must induce Z_3 on $Z_2(S_1/XY)$. So $\langle g \rangle$ is transitive on the three Klein groups of $Z_2(S_1/XY)$ containing $Z(S_1/XY)$. Considering the orders of the centralizers of these subgroups in their action on $XY/Z(XY)$ we obtain a contradiction. We have now shown that $W_{11}^g = W_{11}$. Similarly $W_{21}^g = W_{21}$.

At this point we can complete the proof of the main theorem. We assume condition $(*)$ holds. It suffices to show $G_1G_2 \trianglelefteq G$ and this follows from [16, (2.7)] once the hypotheses of that result are verified. For this we use [16, (2.6)]. Suppose $G_1G_2 \ntrianglelefteq G$. We conclude that there exists a conjugate t_1 of t such that $t_1 \notin N(G_1G_2)$ and $tt_1 \in N(S_i)$, $i = 1, 2$. As in the proof of [16, (2.2)], $O_2(\langle t, t_1\rangle) \leqslant N(G_1G_2)$. So we may assume that $g = tt_1$ has odd order. By (4.6) g

normalizes W_1 and W_2. However, $G_0 = G_1 G_2 = \langle N_G(W_1)^{(\infty)}, N_G(W_2)^{(\infty)} \rangle$, so $G_0 = G_0{}^g$. This contradiction completes the proof of the main theorem.

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